THE GALERKIN METHOD IN ACTION:
A DYNAMIC MODEL FOR THE HANGING STRING

Darren Pais*[1], *Mentor: Brody D. Johnson, PhD

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ABSTRACT. The hanging string is a popular initial-value boundary-value problem (IVPVP) example of using Fourier series to solve partial differential equations. In this paper, we analyze the problem differently, via a finite element approach. The paper focuses on utilizing the Galerkin method to provide a matrix based, dynamic solution, to the hanging string problem. Further, we illustrate the validity of the Galerkin finite element solution by comparisons to more traditional analytical and numerical solutions. In addition, changes in the element mesh parameter are analyzed. We demonstrate that the hanging string problem can be successfully solved by the novel and interesting finite element approach. Further, we suggest that similar methods can be employed in modeling systems that are more complex, for which analytical solutions may be difficult to obtain.

1 INTRODUCTION

1.1 Hanging String

The main aim of this paper is to study the motion of the hanging string using the Galerkin method. In order to build up to this, we initially develop the partial differntial equation and necessary boundary conditions. The string modeled here is a hanging string from a fixed support with a zero slope condition at its lower end. The hanging string is a special case for the general motion of a vibrating string under special boundary conditions. The vibrating string constitutes a simple and important example of a problem involving the wave equation, as in the case of a violin or guitar string. Consider a string under a constant tension T of linear mass density ρ. As an introduction, we first develop the differential equations that describe the motion of the string by considering the force balance on a small element of the string as described in Figure 1. We set up a coordinate system as illustrated in Figure 1 in which the transverse displacement of the string $u(x,t)$ is a function of time t and the lateral position along the string x.

We assume static equilibrium of the string in the $x-$direction, therefore, equal and opposite sforces act on an element of the string in this direction. If we consider the mass density of the string to be small, ρ multiplied by a small length Δx will be much smaller than the tension forces so gravitational forces can be ignored (massless string) giving us,

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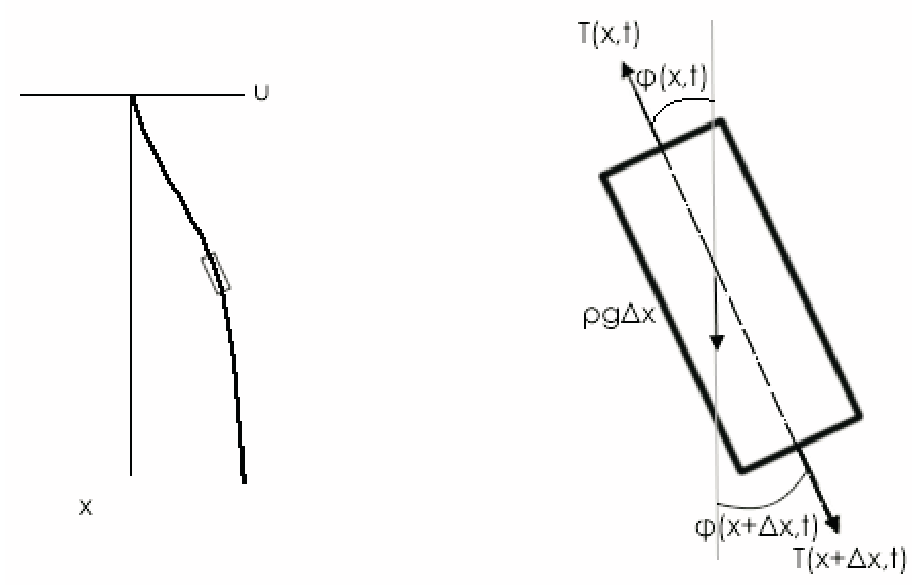

Figure 1: String showing forces acting on a small element.

$$T = T(x + \Delta x, t + \Delta t) \cos \phi(x + \Delta x, t + \Delta t) = T(x, t) \cos \phi(x, t). \tag{1}$$

Here T is the constant magnitude of tension force in the lateral direction. In the $u-$direction, we construct a force balance using Newton's second law including the inertial forces in this direction as follows,

$$T(x + \Delta x, t + \Delta t) \sin \phi(x + \Delta x, t + \Delta t) - T(x, t) \sin \phi(x, t) = m \frac{\partial^2 u}{\partial t^2}(x, t). \tag{2}$$

Here we notice that since $u(x, t)$ represents the displacement of the string in the vertical direction, its second partial derivative with respect to time is the vertical acceleration. Also m represents the mass of a small element of the string ($\rho \Delta x$). Substituting Equation (2) in Equation (1) and noticing that $\tan \phi(x, t)$ can be expressed as the derivative of u with respect to x we have,

$$\frac{T}{\Delta x} \left(\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t) \right) = \rho \frac{\partial^2 u}{\partial t^2}. \tag{3}$$

Taking the limit as $\Delta x \to 0$, the difference quotient tends to the partial derivative with respect to x, leaving Newton's second law in the form,

$$T \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2}. \tag{4}$$

We now make a substitution for c, the speed of wave propagation given by,

$$c = \sqrt{\frac{T}{\rho}}. \tag{5}$$

This provides us with the final equation below to describe the motion of the string,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{6}$$

This equation is the wave equation. Together with the necessary boundary conditions, the wave equation provides a complete simplified analytical description of the vertical hanging string. For our study, the first boundary condition requires that the string remain fixed at the support, i.e. the displacement of the support is 0. The second boundary condition necessitates that the bottom end of the string is hanging freely, i.e. that the slope of the string at its bottom end is 0. These are represented as,

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(a,t) = 0 \tag{7}$$

In order to justify the zero-slope boundary condition at the bottom end of the string, we consider a hypothetical situation in which we have a point mass $'m'$ at the bottom end supported by a spring of spring constant $'k'$ as illustrated in Figure 2.

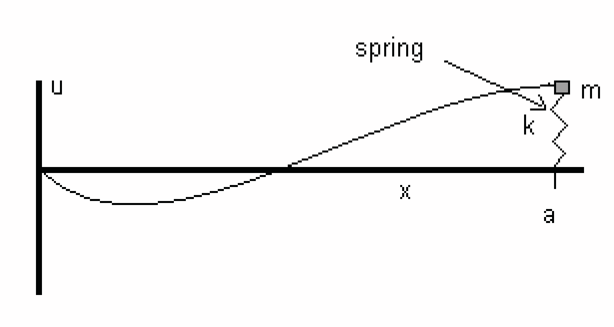

Figure 2: String with point mass $'m'$ at its lower end, connected to spring k.

Applying Newton's second law for forces at the bottom end of the string, we have the relationships

$$T = T(a,t)\cos\phi(a,t), \tag{8}$$

$$m\tfrac{\partial^2 u}{\partial t^2}(a,t) = -T\tan\phi(a,t) - ku(a,t), \tag{9}$$

$$\text{and} \quad \tfrac{\partial u}{\partial x}(a,t) = -\tfrac{1}{T}\left[ku(a,t) + m\tfrac{\partial^2 u}{\partial t^2}(a,t)\right]. \tag{10}$$

We now consider the limiting case of the spring constant and the mass approaching zero in order to obtain the necessary free end boundary condition as,

$$\frac{\partial u}{\partial x}(a,t) = 0 \tag{11}$$

Notice that we have made key assumptions in deriving the wave equation (6), which include, massless string, constant tension and drag free motion. These assumptions help us obtain a simple, yet sufficiently accurate mathematical description of the hanging string for our purposes.

1.2 Methods Employed

We will use three separate methods in order to solve the wave equation for a vertical string under the specified boundary conditions and obtain the motion of the string. The primary method we develop is the *Finite Element Model (FEM)* of the string which involves strong and weak formulations of the problem and the Galerkin method. By this, we model the string as elements of finite length connected to one another via a series of discrete points. In this method, the dynamic motion of the string is reduced to a linear system of ordinary differential equations (ODE's) via the Galerkin approximation. We convert this linear system to its equivalent matrix format and solve to obtain string displacements. However, in order to approach the FEM method, we first use the *Separation of Variables* technique in order to obtain a full analytical solution of the vertical string motion problem. The analytical solution involves representing the dynamic motion of the string using trigonometric functions (Fourier series) and solving for Fourier coefficients. In addition, we use difference equations in order to obtain a *Numerical Replacement Equations* solution. Finally, we compare the results obtained using the three methods in order to validate the Finite Element solution obtained. Each of these solutions in developed in separate sections below. The analytical and numerical solutions are first developed. Major emphasis placed on the FEM solution which is developed right after.

2 SEPARATION OF VARIABLES

2.1 General Description

The initial value - boundary value problem that completely describes the vertical string we intend to consider is described by the following equations,

Partial Differential Equation (PDE):

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad 0 < x < a, \quad t > 0 \tag{12}$$

Boundary Conditions (BC's):

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(a,t) = 0 \tag{13}$$

Initial Conditions (IC's):

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x) \tag{14}$$

Observing the homogeneous PDE and the linear, homogeneous BC's we may apply the method of separation of variables to solve the problem. Thus, we assume a solution for u in the form, $u(x,t) = \phi(x)T(t)$. Our PDE reduces to

$$\phi''(x)T(t) = \frac{1}{c^2}\phi(x)T''(t) \tag{15}$$

Dividing throughout by $\phi(x)T(t)$ we have,

$$\frac{\phi''(x)}{\phi(x)} = \frac{T''(t)}{c^2 T(t)} \tag{16}$$

For this equality to hold, both sides of this equation must be constant as they are functions of independent variables. We say that this constant is $-\lambda^2$ (in order to ensure non-trivial solution) and separate the preceding equation into two ordinary differential equations as,

$$T''(t) + \lambda^2 c^2 T(t) = 0 \tag{17}$$

$$\phi''(x) + \lambda^2 \phi(x) = 0$$

Assuming a solution for ϕ of the form, $\phi(x) = A\cos\lambda x + B\sin\lambda x$ and applying the necessary boundary conditions, for non-trivial solution we have ϕ in the form,

$$\phi_n(x) = \sin(\lambda_n x), \quad \lambda_n = \frac{2n-1}{a}\frac{\pi}{2}, \quad n \in [1,\infty) \tag{18}$$

Now in similar fashion we assume a solution for $T(t)$ for the differential equation for T in Equation (17) of the form, $T_n(t) = a\cos\lambda_n ct + b\sin\lambda_n ct$. The product of the solutions to the differential Equations (17) provides us with linearly independent solutions to our PDE (by separation of variables). By superposition, we can write the general solution as,

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n(x)T_n(t) \tag{19}$$

Writing this solution out in complete form and applying our initial conditions, we obtain the following relationships,

$$u(x,t) = \sum_{n=1}^{\infty} \left[\sin\lambda_n(x)\left\{a_n\cos\lambda_n ct + b_n\sin\lambda_n ct\right\}\right] \tag{20}$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \lambda x \tag{21}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \implies \sum_{n=1}^{\infty} \left\{ \frac{n\pi c}{a} \right\} b_n \sin \lambda_n x = g(x) \tag{22}$$

We now notice that we can extract the Fourier coefficients a_n and b_n in order to obtain the last pieces for our solution to the PDE under the initial and boundary conditions as

$$a_n = \frac{2}{a} \int_0^a f(x) \sin(\lambda_n x) dx \tag{23}$$

$$b_n = \frac{2}{n\pi c} \int_0^a g(x) \sin(\lambda_n x) dx \tag{24}$$

2.2 Specific Initial Conditions

We now attempt to apply specific initial conditions to the analytical solution derived above, in order to study the motion of the string and make comparisons. The initial condition that we choose has the following form,

$$f(x) = \frac{x}{4}, \quad g(x) = 0 \tag{25}$$

This initial condition ensures a linear string position varying from zero displacement (BC) at the node to a maximum amplitude of $\frac{a}{4}$ at its lowest end. Also, we consider the length of the string as 1 unit ($a = 1$). The speed of wave propagation is also set to one unit ($c = 1$). Under these conditions we obtain the following formulas for our Fourier coefficients by substituting and integrating:

$$b_n = 0 \tag{26}$$

$$a_n = \frac{1}{2} \frac{1}{\left[(n - \frac{1}{2})\pi)\right]^2} \sin\left[(n - \frac{1}{2})\pi\right] \tag{27}$$

Note that the initial conditions f and g as well as value assignments for a and c are chosen for computational simplicity in our specific example here. However, the equations presented are general and hence other continuous initial conditions and suitable values for parameters a and c may be used to reach similar conclusions.

3 REPLACEMENT EQUATIONS

3.1 General Description

The replacement equations technique for the solution of a differential equation involves the conversion of the partial derivatives of the differential equation into difference equations. In order to apply this technique to the wave differential Equation (6), we need to designate points $x_i = i\Delta x$ ($\Delta x = \frac{1}{n}$) and times $t_m = m\Delta t$ for which we intend to find the approximation $u(x_i, t_m) \simeq u_i(m)$. According to this designation, our partial derivatives in Equation (6) replaced by their central differences equate to,

$$\frac{\partial^2 u}{\partial x^2} \to \frac{u_{i+1}(m) - 2u_i(m) + u_{i-1}(m)}{(\Delta x)^2} \tag{28}$$

$$\frac{\partial^2 u}{\partial t^2} \to \frac{u_i(m+1) - 2u_i(m) + u_i(m-1)}{(\Delta t)^2} \tag{29}$$

Making difference equation substitutions along with the substitution $\alpha = \frac{c\,\Delta t}{\Delta x}$ in Equation (6) we have the following replacement equation for the wave motion,

$$u_i(m+1) = \alpha^2 u_{i-1}(m) + 2(1-\alpha^2)u_i(m) + \alpha^2 u_{i+1}(m) - u_i(m-1) \tag{30}$$

In order to enforce the boundary condition relationships expressed in Equation (7), we have the following replacement equations for the boundary conditions,

$$\text{Fixed support: } u_0(m) = 0 \tag{31}$$

$$\text{Zero Slope end: } u_{n+1}(m) = u_{n-1}(m) \tag{32}$$

Also, the initial conditions must be satisfied, so the corresponding replacement equations for the initial conditions are,

$$\text{Position IC: } u_i(0) = f(i\Delta x) \tag{33}$$

$$\text{Velocity IC: } \tfrac{u_i(1) - u_i(-1)}{2\Delta t} = g(x_i) \tag{34}$$

3.2 Specific Case

In our example, we will consider a string of length 1 and construct replacement equations dividing the string into 10 units. Additionally, we chose a time step of 0.1 and enforce the initial conditions described in Equation (25) in order to obtain the following parameters for our approximate solution,

$$\begin{gathered} n = 10 \quad \Delta x = 0.1 \quad \alpha = 1 \\ u_i(0) = f(i\Delta x) \quad u_i(1) = u_i(-1) \end{gathered} \tag{35}$$

4 FINITE ELEMENT METHOD

4.1 Definiton

The differential equation form of the problem (wave equation) as introduced in Equation (6), along with its boundary and initial conditions constitute the *Strong Form* of the problem (S). In slightly different notation, where a comma subscript stands for differentiation, this strong form can be expressed as

$$\begin{gathered} u_{,xx} = \frac{1}{c} u_{,tt} \\ \text{BC's}: \; u(0) = 0; \;\; u_{,x}(1) = 0 \\ \text{IC's}: \; u(0, x) = u_0; \;\; u_{,t}(0, x) = \dot{u_0} \end{gathered} \tag{S}$$

We restrict attention to a class of generalized functions which are described in terms of the definition of the L^2 norm of a function f (square-integrable) on the domain,

$$L^2([0,1]) = \left\{ f : [0,1] \to \mathbb{C} : \int_0^1 |f(x)|^2 dx < \infty \right\} \tag{36}$$

We recognize that our solutions need to be continuous and for our purposes we restrict the solution u to confine to the *Hilbert space* of functions which are continuous and piecewise smooth defined as,

$$H^1([0,1]) = \{ u : \; u \in L^2; \; u_{,x} \in L^2 \} \tag{37}$$

The dot product of two functions f and g is assigned new notation as follows,

$$\int_0^1 f(x)\overline{g(x)}dx = \langle f, g \rangle \tag{38}$$

The dot product of the derivatives of two functions f and g is defined as,

$$\int_0^1 f_{,x}(x)\overline{g_{,x}(x)}dx = a\langle f, g \rangle \tag{39}$$

We consider a collection of *trial solutions* ζ consisting of all functions that belong H^1 and satisfy the non-homogeneous boundary condition at $x = a$ as described in Equation (13). Additionally, we define a second class of functions known as the weighting functions ν. The weighting functions are similar to the trial solutions but differ in that they satisfy the homogeneous boundary condition at $x = a$. These are expressed as,

$$(S) = \{\zeta = u \in H^1 \ : \ u_{,x}(a) = 0\} \tag{40}$$
$$(W) = \{\nu = w \in H^1 \ : \ w_a = 0\}$$

Using additional notation expressing the time derivative as $\ddot{u} = \frac{\partial^2 u}{\partial t^2}$ we have the following description of the *Weak Form* of the solution (W) as,

$$a\langle w, u\rangle = -\frac{1}{c^2}\langle w, \ddot{u}\rangle \tag{W}$$

$$\text{IC's:} \quad \langle w, \frac{1}{c}u(t=0)\rangle = \langle w, \frac{1}{c}u_0\rangle; \ \ \langle w, \frac{1}{c}\dot{u}(t=0)\rangle = \langle w, \frac{1}{c}\dot{u_0}\rangle$$

In order to prove that the weak and strong forms of the solution to the PDE are equivalent, we begin with the strong form and simplify to obtain the weak form as follows,

$$u_{,xx} - \frac{1}{c^2}u_{,tt} = 0 \tag{S}$$

Multiplying both sides by w and intergrating over the domain we have,

$$\int_0^a w\frac{\partial^2 u}{\partial x^2}dx - \int_0^a \frac{1}{c^2}w\frac{\partial^2 u}{\partial t^2}dx = 0 \tag{41}$$

Simplifying using integration by parts we have,

$$w(a)u_{,x}(a) - w(0)u_{,x}(0) - a\langle w, u\rangle - \frac{1}{c^2}\langle w, \ddot{u}\rangle = 0 \tag{42}$$

Clearly, the first two terms equate to zero as our solution u and weighting functions w satisfy the $u_{x=0} = 0$ boundary condition and u must satisfy the zero slope at $x = a$ boundary condition.

This leaves us with the initial weak form representation of the solution as in Equation (W), proving that the strong and weak forms are equivalent.

In order to obtain a solution to the weak form of the problem described, we choose a suitable class of shape functions that span our solution space. This set of functions constitutes the *mesh* used to discretize the weak form of the problem. In order to define the mesh we partition our domain ($[0, a]$) into $'n'$ non-overlapping sub-intervals denoted by $[x_A, x_{A+1}]$ where $x_A < x_{A+1}$ and $A = 1, 2, ..., n$. We define our intervals to be of constant length $h = x_{A+1} - x_A$. Here 'h' is known as the *mesh parameter*. Notice that $h = \frac{1}{n}$. We now replace ν and ζ with their finite dimensional subspaces ν^h and ζ^h based on the mesh parameter. Finally, we assume that w^h and u^h can be represented as linear combinations of linearly independent functions N_A with A as defined. Hence we obtain,

$$w^h = \sum_{A=1}^{N} c_A N_A \tag{43}$$
$$u^h = \sum_{B=1}^{N} d_B N_B$$

For our particular problem we chose piecewise linear shape functions N_A, for $2 \leq A \leq n$, as follows,

$$N_A(x) = \begin{cases} \frac{x-x_A}{h}, & x_A \leq x \leq x_{A+1}, \\ \frac{x_{A+2}-x}{h}, & x_{A+1} \leq x \leq x_{A+2}, \\ 0, & \text{elsewhere.} \end{cases} \tag{44}$$

We can easily visualize our chosen shape functions as drawn in Figure 3 and see that they are linearly independent.

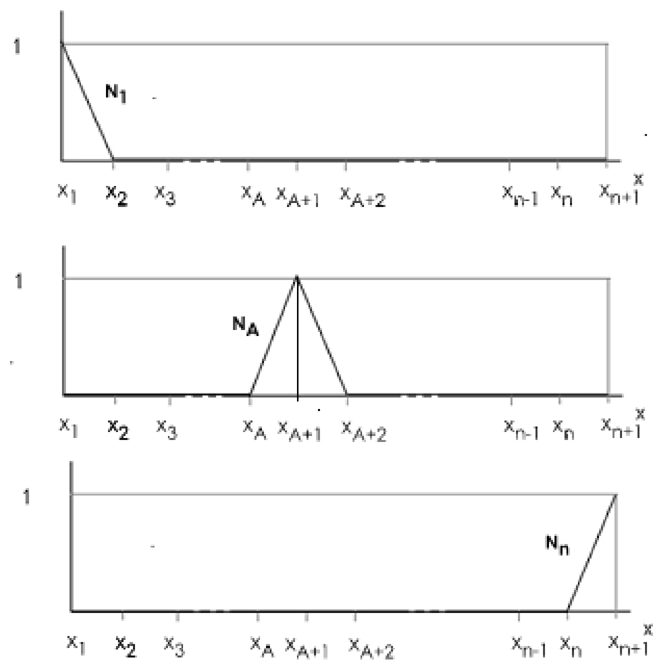

Figure 3: Shape Functions.

Note that our shape functions enforce our boundary condition at $x = 0$. Also note that for a chosen shape parameter, linear combinations of our shape functions span the entire solution subspace ζ. Additionally, each of our shape functions is linearly independent and conforms to the restrictions imposed by Equation (37).

In Equation (44) the parameters c_A and d_B constitute unknown coefficients that determine our solution. In order to calculate these coefficients, we substitute the linear combinations in the weak form of the solution (W), at which point the problem is reduced to a set of linear equations. This is the crux of what is called the *Galerkin Method*.

This substitution yields the following equation,

$$a\langle\sum_{A=1}^{N} c_A N_A, \sum_{B=1}^{N} d_B N_B\rangle = -\frac{1}{c}\langle\sum_{A=1}^{N} c_A N_A, \sum_{B=1}^{N} \ddot{d}_B N_B\rangle \tag{45}$$

Performing similar substitutions in the Initial Conditions of the weak form solution (W) yields the following *Galerkin* form of the problem (G),

$$\begin{aligned} \sum_{B=1}^{N} d_B \; a\langle N_A, N_B\rangle + \sum_{B=1}^{N} \ddot{d}_B \; \frac{1}{c}\langle N_A, N_B\rangle = 0 \\ \sum_{B=1}^{N} d_{Bo} \frac{1}{c} \langle N_A, N_B\rangle = \langle N_A, \frac{1}{c} u_o\rangle \\ \sum_{B=1}^{N} \dot{d}_{Bo} \frac{1}{c} \langle N_A, N_B\rangle = \langle N_A, \frac{1}{c} \dot{u}_o\rangle \end{aligned} \tag{G}$$

The solution to the Galerkin form of the problem arises from expressing (G) in a Matrix form (M) which we will deal with next. To summarize the Finite Element Method, we expressed the strong form of the problem in its equivalent weak form. We then approximated the weak form with the Galerkin approximation, which will give rise to a Matrix form for a solution for our coefficients,

$$(S) \Leftrightarrow (W) \approx (G) \Leftrightarrow (M). \tag{46}$$

4.2 Matrix Solution

The Galerkin form of the solution in represented in Equation (G). Note that in this equation, everything is known except for the d_B's. Thus Equation (G) constitutes a system of n linear second order ordinary differential equations for n unknowns with position and velocity initial conditions for each equation. This can be written in concise form in the following fashion: Let

$$K_{AB} = a\langle N_A, N_B\rangle \tag{47}$$

and

$$M_{AB} = \frac{1}{c}\langle N_A, N_B\rangle, \tag{48}$$

then Equation (G) becomes,

$$\sum_{B=1}^{N} d_B K_{AB} + \sum_{B=1}^{N} \ddot{d_B} M_{AB} = 0, \quad 1 \leq A \leq n \tag{49}$$

Further simplification is gained by adopting a matrix notation. Let

$$\mathbf{K} = [K_{AB}] = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ K_{n1} & \cdots & K_{n(n-1)} & K_{nn} \end{bmatrix}, \tag{50}$$

$$\mathbf{M} = [M_{AB}] = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ M_{n1} & \cdots & M_{n(n-1)} & M_{nn} \end{bmatrix}, \tag{51}$$

and

$$\mathbf{d} = [d_B] = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}. \tag{52}$$

Now Equation (49) may be written as

$$[\mathbf{M}] \left[\ddot{\mathbf{d}} \right] + [\mathbf{K}] [\mathbf{d}] = 0 \tag{53}$$

The matrix **K** is often refered to as the stiffness matrix, **M** as the mass matrix and **d** as the displacement vector. It is interesting to notice that Equation (53) takes the form of a linear system of differential equation representing the displacements and forces of a spring-mass system in mechanical systems.

At this point we can state the matrix equivalent (M) of the Galerkin problem:

$$\begin{gathered} \text{Given Matrices } \mathbf{M} \text{ and } \mathbf{K}\text{, find } \mathbf{d} \text{ such that, } [\mathbf{M}] \left[\ddot{\mathbf{d}} \right] + [\mathbf{K}] [\mathbf{d}] = 0 \\ \text{With initial conditions, } [d_{Bo}] = \frac{1}{c} \mathbf{M}^{-1} \langle N_B, u_o \rangle \\ \left[\dot{d_{Bo}} \right] = \frac{1}{c} \mathbf{M}^{-1} \langle N_B, \dot{u_o} \rangle \end{gathered} \tag{M}$$

By examining the matrix form of the solution, we can decouple the second order differential equation into two first order differential equations,

$$\frac{d}{dt}\mathbf{d} = \dot{\mathbf{d}} \tag{54}$$

$$\frac{d}{dt}\dot{\mathbf{d}} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{d}$$

Expressing this as a single matrix equation in concise notation we have,

$$\frac{d}{dt}\begin{bmatrix} d_1 \\ \vdots \\ d_n \\ \dot{d}_1 \\ \vdots \\ \dot{d}_n \end{bmatrix} = \begin{bmatrix} [0]_{nxn} & [I]_{nxn} \\ \\ \left[-\mathbf{M}^{-1}\mathbf{K}\right]_{nxn} & [0]_{nxn} \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \\ \dot{d}_1 \\ \vdots \\ \dot{d}_n \end{bmatrix} \tag{55}$$

Equation (55) is a matrix differential equation of the form $\left[\dot{X}\right] = [A]\,[X]$ which has the solution of the form $[X]\,(t) = [X_0]\,e^{[A]t}$ where t represents the time at which we want the solution, $[X_0]$ is the initial conditions vector and $e^{[A]t}$ is the matrix exponential which is given by the *Taylor series* expansion,

$$e^{[A]t} = I + \frac{[A]\,t}{1!} + \frac{([A]\,t)^2}{2!} + \frac{([A]\,t)^3}{3!} + \cdots . \tag{56}$$

We are now left with the task of determining the contents of matrices $\mathbf{M}$ and $\mathbf{K}$ and the coefficients d_0 and $\dot{d}_0$ corresponding to our initial conditions $f(x)$ and $g(x)$ respectively. In order to do so we consider the expression $\langle N_x, N_x\rangle$ where $1 \leq x \leq n-1$ for shape functions as described in Figure 3,

$$\langle N_x, N_x\rangle = \int_0^1 N_x.N_x\,dx = 2\int_0^h \frac{x^2}{h^2}dx = \frac{2}{3}h \tag{57}$$

Performing similar integrals, we obtain the following expressions

$$\langle N_x, N_{x+1}\rangle = \frac{h}{6} \quad \text{where, } x = 1,\ 2, \cdots,\ n-1 \tag{58}$$

and

$$\langle N_n, N_n\rangle = \frac{h}{3} \tag{59}$$

Thus, the mass matrix takes the form

$$[\mathbf{M}] = \frac{1}{c}\begin{bmatrix} \frac{2}{3}h & \frac{h}{6} & 0 & 0 & \cdots & 0 \\ \frac{h}{6} & \frac{2}{3}h & \frac{h}{6} & 0 & \cdots & 0 \\ 0 & \frac{h}{6} & \frac{2}{3}h & \frac{h}{6} & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & \frac{h}{6} & \frac{2}{3}h & \frac{h}{6} \\ 0 & 0 & \cdots & \cdots & \frac{h}{6} & \frac{h}{3} \end{bmatrix} \tag{60}$$

Notice that the mass matrix obtained in Equation (60) is banded and also symmetric, i.e., $\mathbf{M}^T = \mathbf{M}$. We now consider the elements of the stiffness matrix,

$$a\langle N_x, N_x\rangle = \int_0^1 (N_x)'.(N_x)'\,dx = 2\int_0^h \frac{1}{h}^2 dx = \frac{2}{h}, \tag{61}$$

$$a\langle N_x, N_{x+1}\rangle = \frac{-1}{h}, \quad \text{where, } 1 \le x \le n-1 \tag{62}$$

and

$$a\langle N_n, N_n\rangle = \frac{1}{h}. \tag{63}$$

The stiffness matrix takes the form

$$[\mathbf{K}] = \begin{bmatrix} \frac{2}{h} & \frac{-1}{h} & 0 & 0 & \cdots & 0 \\ \frac{-1}{h} & \frac{2}{h} & \frac{-1}{h} & 0 & \cdots & 0 \\ 0 & \frac{-1}{h} & \frac{2}{h} & \frac{-1}{h} & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & \frac{-1}{h} & \frac{2}{h} & \frac{-1}{h} \\ 0 & 0 & \cdots & \cdots & \frac{-1}{h} & \frac{1}{h} \end{bmatrix} \tag{64}$$

The initial conditions can be solved for by substitution in the Equations of (M). With the definitions in this section, we have a complete description of the solution to the Finite Element Model of the vertical string. We will now apply this solution using our chosen specific initial conditions and obtain a particular solution in the following section.

4.3 Specific Case Solution

In our specific case, the initial conditions are as stated in Equation (25),

$$f(x) = \frac{x}{4}, \quad g(x) = 0 \tag{65}$$

Corresponding to our displacement initial condition $f(x)$, we obtain coefficients d_{Bo} using Equation (M). In order to do this we obtain integrals,

$$\langle N_B, u_0 \rangle = \int_{(B-1)h}^{Bh} f(x)\frac{1}{h}x - (n-1)h \; dx + \int_{Bh}^{(B+1)h} f(x)\frac{-1}{h}x - (n+1)h \; dx \qquad (66)$$
$$= \frac{1}{4}h^2 B \qquad \text{For } B = 1, 2, \cdots, (n-1)$$

$$\langle N_n, u_0 \rangle = \int_{(n-1)h}^{nh} f(x)\frac{1}{h}x - (n-1)h \; dx \qquad (67)$$
$$= \frac{1}{4}\left[\frac{nh^2}{2} - \frac{h^2}{6}\right]$$

We take our velocity initial condition function $g(x)$ to be 0 and hence all our corresponding $\dot{d}_0$ coefficients will be zero.

With the integral evaluations listed above substituted in our matrix form Equation (M), we can obtain necessary coefficients for the initial conditions and solve the matrix equation Equation (55).

5 COMPARISONS OF RESULTS

5.1 Graphical Comparisons

In the previous sections, we described an Analytical, Numerical, and Finite Element solutions using the Galerkin approximation, to the vertical hanging string problem with particular emphasis on the Finite Element solution. In this section, we look at the relative quality of the finite element solution by comparing it to the analytical and numerical solutions. We graphically visualize the finite element solution obtained for a string of length 1 unit, initial conditions $f(x)$ and $g(x)$ as earlier specified in Equation (25), and a mesh parameter h of 0.1.

It is important to note here that the various figures to follow represent solutions for a relatively large mesh parameter of 0.1 (one-tenth of a string length of 1 unit). Smaller mesh parameters will yield better solutions as we increase the degrees of freedom, the number of simultaneous differential equations and the number of points in our solution, by shortening the mesh parameter.

In Figure 4, we plot the FEM solution at various times in order to visulize the hanging string solution obtained. As we can see in Figure 4, the Finite element model solution of the string enforces the necessary boundary conditions and initial conditions and also appears to simulate the actual motion of a hanging vertical string. Further, we superimpose the displacements of the string at various times in Figure 5 in order to get a comprehensive idea of the string vibration. Figure 5 further illustrates that the FEM solution obtained appears to conform to our expecation of hanging string motion for the given initial conditions.

Figure 6 is a visual comparison of the solutions obtained comparing the displacements at $t = 1.5$. This figure provides the first concrete validation of the FEM solution. We notice in Figure 6 that our FEM solution conforms quite well to the analytical and numerical solutions.

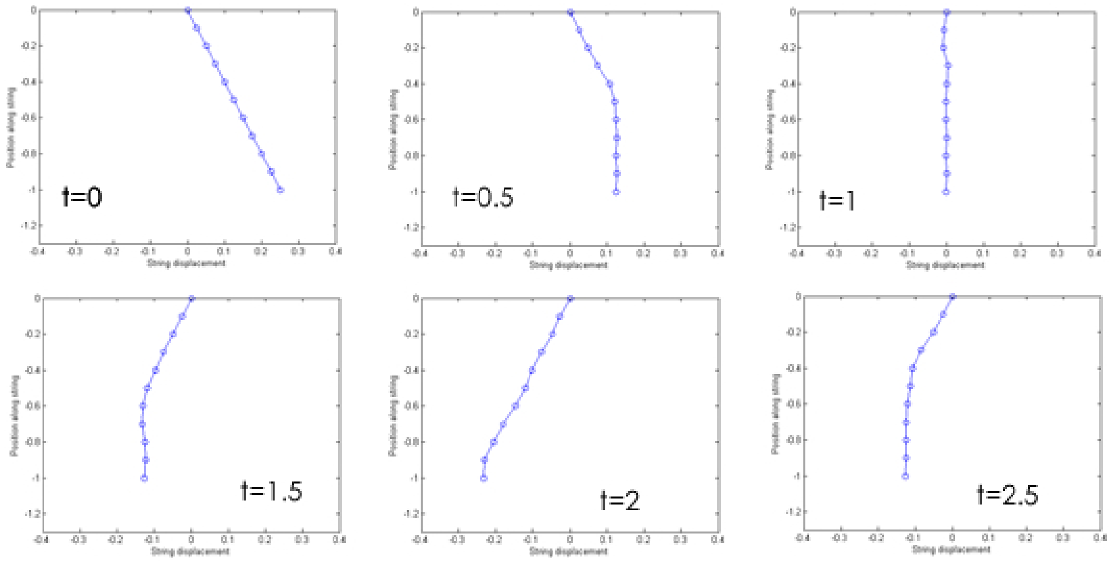

Figure 4: Finite element solution showing at various times.

5.2 Numerical Comparisons

We will now compare the displacements at sample locations ($x = 0.1n$, $0 \leq n \leq 1$) for the Analytical, Numerical and FEM Galerkin approximation solution at various times.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
u_A	0	0.0250	0.0500	0.0750	0.1000	0.1247	0.1250	0.1250	0.1250	0.1250	0.1250
u_F	0	0.0254	0.0491	0.0755	0.0966	0.1183	0.1301	0.1326	0.1248	0.1223	0.1256
u_N	0	0.0250	0.0500	0.0750	0.1000	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250

Table 1: Displacements at t=1.5, u_A: Analytical, u_F: FEM, u_N: Numerical.

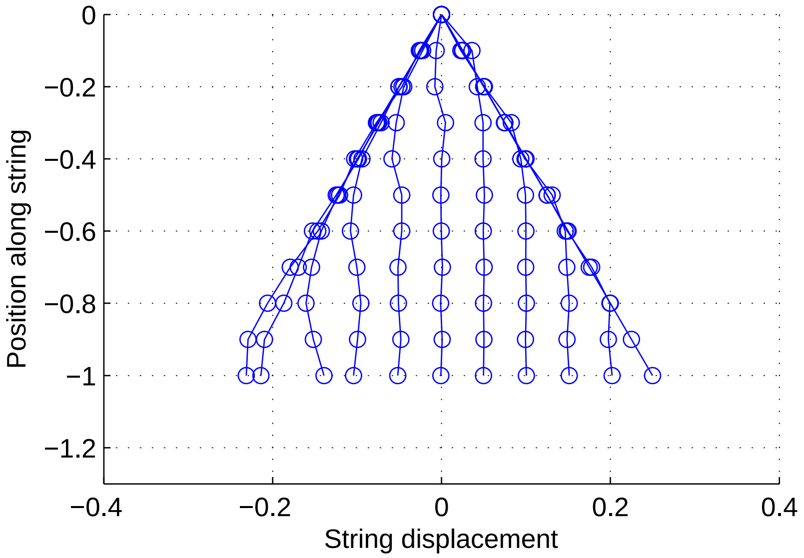

Figure 5: Comprehensive plot of displacements for times $t_n = 0.2n$, $0 \leq n \leq 10$.

For a mesh parameter $h = 0.1$, and times $t = 1.5,\ 4$ the results are tabulated in Tables 1 and 2.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
u_A	0	0.0250	0.0500	0.0750	0.1000	0.1250	0.1500	0.1750	0.2000	0.2250	0.2495
u_F	0	0.0292	0.0535	0.0701	0.0983	0.1213	0.1505	0.1841	0.2087	0.2223	0.2299
u_N	0	0.0250	0.0500	0.0750	0.1000	0.1250	0.1500	0.1750	0.2000	0.2000	0.2000

Table 2: Displacements at t=4, u_A: Analytical, u_F: FEM, u_N: Numerical.

From Tables 1 and 2 we can observe that the FEM Galerkin solution obtained conforms well to the analytical solution indicating the high quality of the Galerkin solution and the overall success in using the FEM method to model a hanging string. Also, in Table 2 ($t = 4$) we notice that the FEM solution conforms to the theoretical solution better than the corresponding numerical approximation (look at $x = 0.8,\ 0.9,\ 1.0$).

In Table 3 below we compare the quality of the FEM solution for different mesh parameters $h = 0.1$ and $h = 0.01$ at $t = 4$,

In Table 3, we can clearly see that by decreasing the mesh parameter by a factor of 10, we greatly improve the quality of our FEM solution, thereby illustrating that the selection of an appropriate mesh parameter is essential in determining the precision we require from our FEM solution. It is also important to note here that smaller mesh parameters imply more degrees of freedom, larger

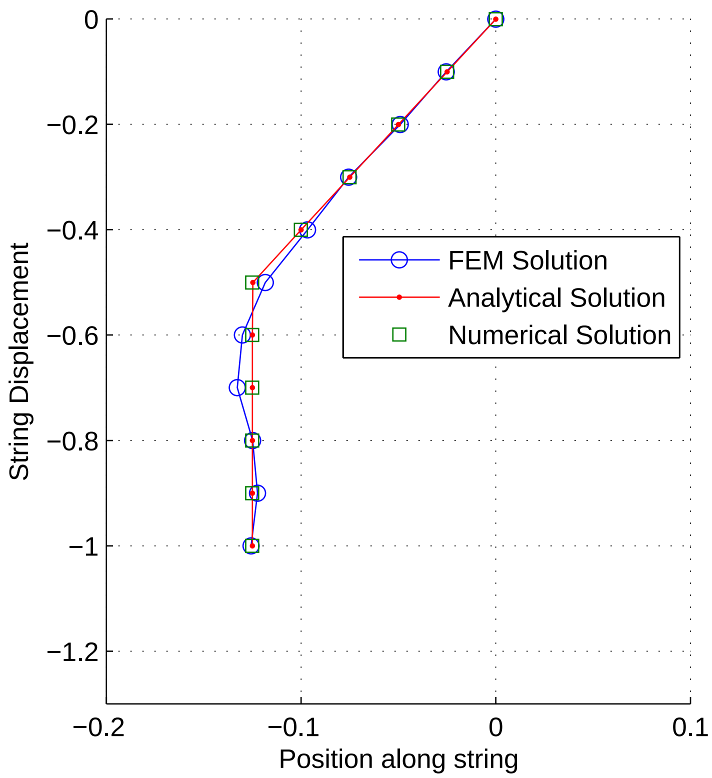

Figure 6: Comparison of solution at t=0.5.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
u_A	0	0.0250	0.0500	0.0750	0.1000	0.1250	0.1500	0.1750	0.2000	0.2250	0.2495
u_{10}	0	0.0292	0.0535	0.0701	0.0983	0.1213	0.1505	0.1841	0.2087	0.2223	0.2299
u_{100}	0	0.0249	0.0500	0.0749	0.1000	0.1251	0.1498	0.1748	0.2002	0.2246	0.2452

Table 3: Displacements at t=4, u_A: Analytical, u_{10}: h=0.1, u_{100}: h=0.01.

matrices and therefore more computational complexity.

6 CONCLUSIONS

The partial differential equation describing the vertical hanging string problem (wave equation) along with its corresponding boundary conditions were derived by applying Newton's law to a vertical string element. An analytical solution to the hanging string problem was obtained by separation of variables and solving second order differential equations for Fourier coefficients. A numerical solution to the problem was also determined using difference equations that replace the partial derivatives of the wave equation and boundary conditions. The main focus of this report was the development of the Finite Element Model (FEM) of the problem and its solution using the Galerkin method/approximation. By this method, the problem is reduced to a linear system of ordinary differential equations that were solved using matrices to obtain string displacements. The results of the three separate methods employed were compared, illustrating that the FEM solution conformed well to the analytical solution of the problem. Further, comparisons between FEM solutions using different mesh parameters illustrated improvement in the solution on decreasing the mesh parameter. Finally, we conclude that the Galerkin method gives us a novel and interesting way to solve the hanging string problem. Since comparisons with the analytical solution indicate the validity of the Galerkin solution, we conclude that this method can be employed in modeling more complex systems whose analytical solutions might be difficult to determine.

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Darren Pais, Aerospace Engineering & Mathematics (Applied), Class of 2007, Saint Louis University.

MSC 2262
20 N. Grand Boulevard
Saint Louis, MO, 63103

paisd @ slu dot edu