FOURIER ANALYSIS[1]

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1 INTRODUCTION

This section provides brief descriptions of various key definitions and formulas that will be used throughout the report.

1.1 Dot Product

Consider the space $\mathbb{R}^3$ compsising all of three dimensional space. The dot product of two vectors $\vec{v}$ and $\vec{w}$ in this space as shown in the figure below is given by,

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{3} v_i w_i \tag{1}$$

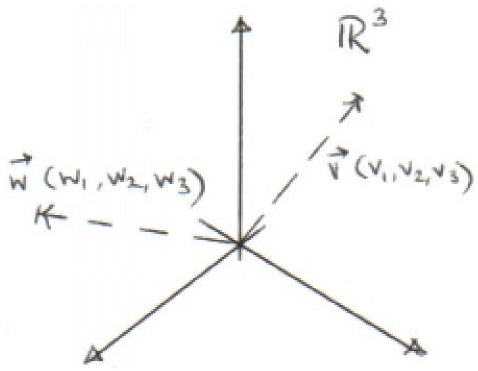

Figure 1: 2 vectors in $\mathbb{R}^3$

Here v_i and w_i represent the projections of vectors $\vec{v}$ and $\vec{w}$ respectively onto the orthogonal vectors that span $\mathbb{R}^3$. This concept of dot product can be further extended to a collection of vectors with a basis of rank N as follows,

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{N} v_i w_i \tag{2}$$

[1]Report designed using LaTeX.

1.2 Orthogonality And Orthonormality

The Basis of a vector space is a linearly independent set that spans the vector space. Suppose we have an n-dimensional vector space with basis $X_1, ..., X_n$ and that a dot product is defined on this vector space. This basis is considered an Orthogonal Basis if for all elements of the basis,

$$X_i \cdot X_j = 0 \text{ if } i \neq j \tag{3}$$

The basis $\{X_1, ..., X_n\}$ will be an Orthonormal basis if it is orthogonal and $X_i \cdot X_i = \|X_i\|^2 = 1$ for $i = 1, ..., n$.

Consider the space $\mathbb{R}^3$ comprising all of three dimensional space. The unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ such that $\hat{\imath} \perp \hat{\jmath} \perp \hat{k}$ spans all of $\mathbb{R}^3$ and therefore constitutes an orthonormal basis for $\mathbb{R}^3$. The dot product of two vectors $\vec{v}$ and $\vec{w}$ in this space was illustrated in the previous section.

Next we consider the dot product of two continuous Riemann integrable functions f and g. If the functions exist in the domain $[a, b]$, their dot product, i.e. $f \cdot g$ can be expressed as the Riemann sum of the products of the values obtained by sampling each function at regular intervals, with a total number of intervals of N. This in equation form is,

$$f \cdot g = \sum_{i=1}^{N} f((i-1)\Delta x)g((i-1)\Delta x)\Delta x \tag{4}$$

Limiting N to approach infinity, the Riemann sums equate to,

$$f \cdot g = \int_a^b f(x)g(x)dx \tag{5}$$

In work in this report we will often take $a = -\pi$ and $b = \pi$, thus limiting our domain to the torus which is represented by the symbol $\mathbb{T}$. Note also that alternative notation for the dot product of f and g is $\langle f, g\rangle$, which is also used in this text. We will also define $\ell^p(\mathbb{Z})$ as a set of coefficients a_k such that the summation series of $\|a_k\|^p$ converges,

$$\ell^p(\mathbb{Z}) = \{a_k\} : \sum_{k\in\mathbb{Z}} \|a_k\|^p < \infty \tag{6}$$

A similar definition is made for functions f as follows,

$$L^p(\mathbb{T}) = \left\{ f, 2\pi - \text{periodic} : \|f(x)\|^p = \int_{\mathbb{T}} \|f(x)\|^p dx < \infty \right\} \tag{7}$$

A graphical description of the funtions $f(x)$ and $g(x)$ is provided below.

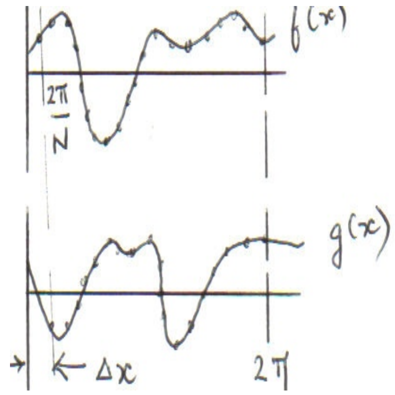

Figure 2: Functions $f(x)$ and $g(x)$ at a regular sampling interval of $\frac{2\pi}{N}$

1.3 Orthogonality Relationships

We now study various relationships between trigonometric sine and cosine functions to check and see under what conditions the functions are orthogonal to each other based on the definition of orthogonality described earlier. The functions looked at will be the $\sin(mx)$ and $\cos(nx)$ funtions, where $n, m \in \mathbb{N}$.

I)

$$\langle \cos(nx), \cos(mx) \rangle = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx$$

Formula : $\cos C + \cos D = 2\cos(\frac{C+D}{2})\cos(\frac{C-D}{2})$

if $nx = \frac{C+D}{2}$ and $mx = \frac{C-D}{2}$, then, $C = (m+n)x$ and $D = (n-m)x$

thus,

$$\begin{aligned}\langle \cos(nx), \cos(mx) \rangle &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((m+n)x) + \cos((n-m)x)dx \\ &= \frac{1}{2}\left[\frac{\sin((m+n)x)}{(m+n)} + \frac{\sin((n-m)x)}{(n-m)}\right]_{-\pi}^{\pi}\end{aligned}$$

thus,

$$\langle \cos(nx), \cos(mx) \rangle = \begin{cases} 0, & n \neq m, \\ \pi, & n = m \neq 0, \\ 2\pi, & n = m = 0. \end{cases} \tag{8}$$

II)

$$\langle \cos(nx), \sin(mx) \rangle = \int_{-\pi}^{\pi} \cos(nx) \sin(mx) dx$$

Formula : $\sin C - \sin D = 2\cos(\frac{C+D}{2})\sin(\frac{C-D}{2})$

if $nx = \frac{C+D}{2}$ and $mx = \frac{C-D}{2}$, then, $C = (m+n)x$ and $D = (n-m)x$

thus,

$$\begin{aligned} \langle \cos(nx), \sin(mx) \rangle &= \frac{1}{2} \int_{-\pi}^{\pi} \sin((m+n)x) - \sin((n-m)x) dx \\ &= \frac{1}{2} \left[\frac{-\cos((m+n)x)}{(m+n)} + \frac{\cos((n-m)x)}{(n-m)} \right]_{-\pi}^{\pi} \end{aligned}$$

thus,

$$\langle \cos(nx), \cos(mx) \rangle = 0 \text{ since, } \cos(p\pi) = 0 \text{ when, } p \in \mathbb{N} \tag{9}$$

III)

$$\langle \sin(nx), \sin(mx) \rangle = \int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx$$

Formula : $\cos C - \cos D = 2\sin(\frac{C+D}{2})\sin(\frac{D-C}{2})$

if $nx = \frac{C+D}{2}$ and $mx = \frac{D-C}{2}$, then, $C = (n-m)x$ and $D = (n+m)x$

thus,

$$\langle \sin(nx), \sin(mx) \rangle = \frac{1}{2} \int_{-\pi}^{\pi} \cos((n-m)x) - \cos((n+m)x) dx$$

$$= \frac{1}{2}\left[\frac{\sin((n-m)x)}{(n-m)} - \frac{\sin((n+m)x)}{(n+m)}\right]_{-\pi}^{\pi}$$

thus,

$$\langle \sin(nx), \sin(mx) \rangle = \begin{cases} 0, & n \neq m, \\ 0, & n = m = 0, \\ \pi, & n = m \geq 1. \end{cases} \tag{10}$$

IV)

$$\langle 1, \sin(nx) \rangle = \int_{-\pi}^{\pi} \sin(nx)dx$$

thus, $\langle$ 1,$\sin(nx)$ $\rangle$ = $\frac{-\cos(nx)}{n}\Big|_{-\pi}^{\pi}$
thus,

$$\langle 1, \sin(nx) \rangle = 0 \tag{11}$$

V)

$$\langle 1, \cos(nx) \rangle = \int_{-\pi}^{\pi} \cos(nx)dx$$

thus, $\langle$ 1,$\cos(nx)$ $\rangle$ = $\frac{\sin(nx)}{n}\Big|_{-\pi}^{\pi}$
thus,

$$\langle 1, \cos(nx) \rangle = 0 \tag{12}$$

1.4 Convergence Of Functions

This section describes ideas of convergence for sequences of functions and distinguishes between pointwise and uniform convergence for functions. Ideas of convergence developed in this section will be utilized further on to describe the convergence properties of Fourier series approximations.

A) POINTWISE CONVERGENCE: For each $n = 1, 2, ...$, let f_n be a function defined on a subset of real numbers. The sequence $\{f_n(x)\}_n$ converges to $g(x)$ *pointwise* if and only if $\lim_{n\to\infty} f_n(x) = g(x)$ for each x. This pointwise convergence will be true if for any given x_0 and any chosen $\epsilon > 0$, $\exists$ $N(x_0) > 0$ such that $n > N(x_0) \Rightarrow \|f_n(x_0) - g(x_0)\| < \epsilon$.

B) UNIFORM CONVERGENCE: For functions $f_n(x)$ and $g(x)$ as described above, we say that $\lim_{n\to\infty} f_n(x) = g(x)$ uniformly if for any $\epsilon > 0$, $\exists\ N > 0$ such that $n \geq N$ implies $\|f_n(x) - g(x)\| < \epsilon$, $\forall\ x$.

The crucial difference between pointwise and uniform convergence is that in uniform convergence we are able to chose a number $N(\epsilon)$ that works for every x. This is not necessarilly possible in the case of pointwise convergence. The figure below clearly illustrates this. In the uniform limit, if we draw a belt extending an amount ϵ on either side of the function f, the for $n > N(\epsilon)$,$f_n(x)$ must be within this belt for every x.

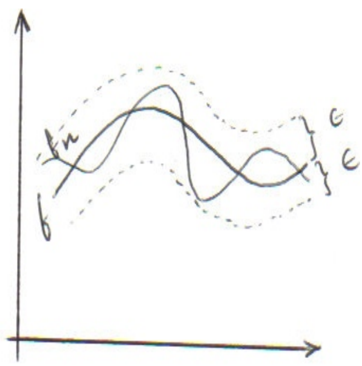

Figure 3: Example of uniform convergence

2 FOURIER SERIES

In this section, the Fourier Series is defined and some of its features are studied.

2.1 Definition of Fourier Series

This section utilizes the ideas developed in section 1 in order to arrive at a series expansion of a given function f(x). This series is titled the Fourier Series. The orthogonality relationships developed are particularly used in order to arrive at the series. Further, ideas of convergence of Fourier series will be explored.

assume

$$f(x) = \tilde{a}_0 + \sum_{n=0}^{N}(a_n \cos nx + b_n \sin nx)$$

Then,

$$\begin{aligned}\langle f, \cos(kx)\rangle &= \int_{-\pi}^{\pi} f(x) \cos kx dx \\ &= \int_{-\pi}^{\pi} a_0 \cos(kx)dx + \int_{-\pi}^{\pi} a_1 \cos(x)\cos(kx)dx + \int_{-\pi}^{\pi} b_1 \sin(x)\cos(kx)dx + \dots \\ &\dots + \int_{-\pi}^{\pi} a_N \cos(Nx)\cos(kx)dx + \int_{-\pi}^{\pi} a_N \sin(Nx)\cos(kx)dx\end{aligned}$$

From orthogonality relationships derived earlier, all but one term reduce to 0, therefore,

$$= \int_{-\pi}^{\pi} a_k \cos(kx)\cos(kx)dx$$

thus,

$$\langle f, \cos(kx)\rangle = \begin{cases} a_k\pi, & k \neq m, \\ 2a_k\pi, & k = 0. \end{cases} \tag{13}$$

$$\langle f, \sin(kx)\rangle = \begin{cases} b_k\pi, & k \in N. \end{cases} \tag{14}$$

Which leads us to,

$$a_k = \tfrac{1}{\pi} \langle f, \cos(kx)\rangle$$
$$b_k = \tfrac{1}{\pi} \langle f, \sin(kx)\rangle$$
$$\text{and } \tilde{a}_0 = \tfrac{a_0}{2}$$

If we consider $N \to \infty$, we obtain the following summary of Fourier Series Formulae,

Fourier Series Formulae

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad n \geq 0 \tag{15}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \quad n \geq 1 \tag{16}$$

$$a_0 = \frac{1}{2\pi}\int_{0}^{2\pi} f(x)dx \tag{17}$$

Fourier Series for $f(x)$:-

$$f(x) \mapsto \frac{a_0}{2} + \sum_{n=0}^{\infty}(a_n \cos(nx) + b_n \sin(nx)) \tag{18}$$

2.2 Quality of Fourier Series

In this section we describe a proof that demonstrates that a function description using Fourier coefficients is the best possible approximation by comparing the error we obtain while using Fourier Coefficients with that obtained by using arbitrary functions.

Let $\{\phi_1, \phi_2, \phi_3, \ldots, \phi_n\}$ be an orthonormal set of functions and let $\{c_1, c_2, ..., c_n\}$ be the Fourier coefficients for a funtion $f \in F$ with respect to this set (where F represents all 2π periodic piecewise continuous functions), i.e.

$$c_i = \int_{-\pi}^{\pi} f(x)\phi_i(x)dx = \langle f, \phi_i \rangle \quad i = 1, \ldots, n \tag{19}$$

Let d_i, $1 \leq i \leq n$ be another sequence of arbitrarily chosen reals. We define two summation sequences as follows,

$$S_n(x) = \sum_{i=1}^{n} c_i\phi_i(x) \tag{20}$$

and,

$$T_n(x) = \sum_{i=1}^{n} d_i\phi_i(x) \tag{21}$$

To prove that the Fourier series constitutes the best approximation, it will be shown that the L^2 norm of the error is smaller for S_n than for T_n. In other words we will show that,

$$\langle f - S_n, f - S_n \rangle \leq \langle f - T_n, f - T_n \rangle \tag{22}$$

If equality holds, $c_i = d_i \;\; \forall \;\; i$. We now prove this result to verify the theorem. We first look at the L^2 norm for T_n,

$$\begin{aligned}
\langle f - T_n, f - T_n \rangle &= \int_{-\pi}^{\pi} \left. f(x) - \sum_{i=1}^{n} d_i\phi_i(x) \right.^{2} dx \\
&= \int_{-\pi}^{\pi} (f(x))^2 - 2f(x)\sum_{i=1}^{n} d_i\phi_i(x) + (\sum_{i=1}^{n} d_i\phi_i(x))^2 dx \\
&= \langle f, f \rangle - 2\sum_{i=1}^{n} d_i c_i + \sum_{i=1}^{n} d_i^2 \\
&= \langle f, f \rangle - \sum_{i=1}^{n} c_i^2 + \sum_{i=1}^{n} c_i^2 - 2\sum_{i=1}^{n} d_i c_i + \sum_{i=1}^{n} d_i^2
\end{aligned}$$

$$= \langle f, f\rangle - \sum_{i=1}^{n} c_i^2 + \sum_{i=1}^{n} (c_i - d_i)^2 \tag{23}$$

We now look at the L^2 norm for S_n and compare it to the L^2 norm of T_n,

$$\begin{aligned}
\langle f - S_n, f - S_n\rangle &= \int_{-\pi}^{\pi} f(x) - \sum_{i=1}^{n} c_i \phi_i(x)^{2} \, dx \\
&= \langle f, f\rangle - 2\sum_{i=1}^{n} c_i^2 + \sum_{i=1}^{n} c_i^2 \\
&= \langle f, f\rangle - \sum_{i=1}^{n} c_i^2
\end{aligned} \tag{24}$$

Comparing equations 23 and 24 we can conclude that,

$$\langle f - T_n, f - T_n\rangle = \langle f - S_n, f - S_n\rangle + \sum_{i=1}^{n} (c_i - d_i)^2 \tag{25}$$

The above equation clearly proves what we set out to prove in equation 22 since the last sum is strictly non-negative.

2.3 Bessel's Inequality

This inequality is derived from the inequality established in equation 24. We notice $\langle f - S_n, f - S_n\rangle \geq 0$ since it represents a dot product of two like terms. Thus, we have the following relation,

$$\begin{aligned}
\langle f - S_n, f - S_n\rangle &\geq 0 \\
\langle f, f\rangle - \sum_{i=1}^{n} c_i^2 &\geq 0 \\
\langle f, f\rangle &\geq \sum_{i=1}^{n} c_i^2
\end{aligned} \tag{26}$$

Equation 26 above is Bessel's inequality. For the orthogonal basis defined earlier, $\{1, \cos x, \sin x, \ldots, \cos(nx), \sin nx\}$,

$$\|\cos nx\|^2 = \|\sin nx\|^2 = \pi, \text{ for n} \geq 1 \tag{27}$$

We have defined Fourier Coefficients for this orthogonal basis to be,

$$a_o = \frac{1}{\pi}\langle f, 1\rangle, \quad a_n = \frac{1}{\pi}\langle f, \cos nx\rangle, \quad b_n = \frac{1}{\pi}\langle f, \sin nx\rangle \tag{28}$$

Coefficients for the corresponding orthonormal system would be, $\frac{a_o\sqrt{\pi}}{\sqrt{2}}, a_n\sqrt{\pi}, b_n\sqrt{\pi}$.

Applying Bessel's inequality to this normalized basis we now have,

$$\pi\left\{\frac{a_o^2}{2} + \sum_{i=1}^{n}(a_i^2 + b_i^2)\right\} \leq \|f\|^2$$
$$\frac{a_o^2}{2} + \sum_{i=1}^{n}(a_i^2 + b_i^2) \leq \frac{1}{\pi}\|f\|^2 \tag{29}$$

2.4 Riemann - Lebesgue Lemma

This lemma uses the results earlier obtained to prove that the terms a_n and b_n earlier defined, converge to 0 as $n \to \infty$.

To prove this we first realize that,

$$\sum_{i=1}^{\infty} a_i^2 \leq \sum_{i=1}^{\infty} a_i^2 + b_i^2 \tag{30}$$

From the results of Bessel's inequality earlier derived in equation 29,

$$a_0^2\sum_{i=1}^{\infty} a_i^2 + b_i^2 \leq \frac{1}{\pi}\|f\|^2 \leq \infty \tag{31}$$

This holds true since the integral to the right convereges. Thus, since

$$\sum_{i=1}^{\infty} a_i^2 < \infty$$

i.e. the sum converges, then, $\lim_{n\to\infty} a_n = 0$. We can show the same holds true for b_n by applying a similar procedure.

2.5 Visual Understanding

In this section we attempt to visualize some of the Fourier relationships earlier derived. We will attempt to construct a Fourier approximation for the funtion $f(x)$ over the domain -2 to 2. We will first look at the Fourier approximation with 1 term and subsequently increase the number of terms to observe a better approximation for a higher order Fourier series.

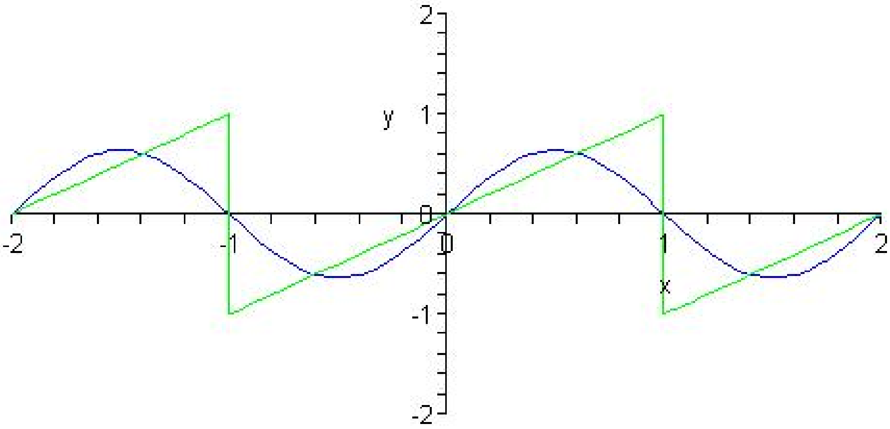

Figure 4: First order Fourier approximation for f(x)=x

The function $f(x)$ chosen above is the $f(x) = x$ function shifted by 2 units at intervals of 2 units. We now look at the approximation when we increase the number of terms to 5. We can see that the Fourier series approximation becomes better[2].

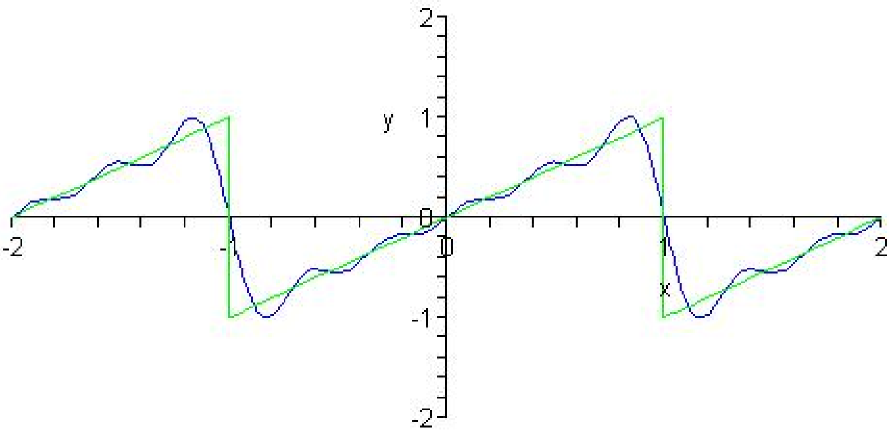

Figure 5: 5th order Fourier approximation for f(x)=x

[2]Graphs constructed in Maplesoft MAPLE 9.5.

We now increase the number of terms to 20 and as we can clearly see the function approximation becomes nearly exact.

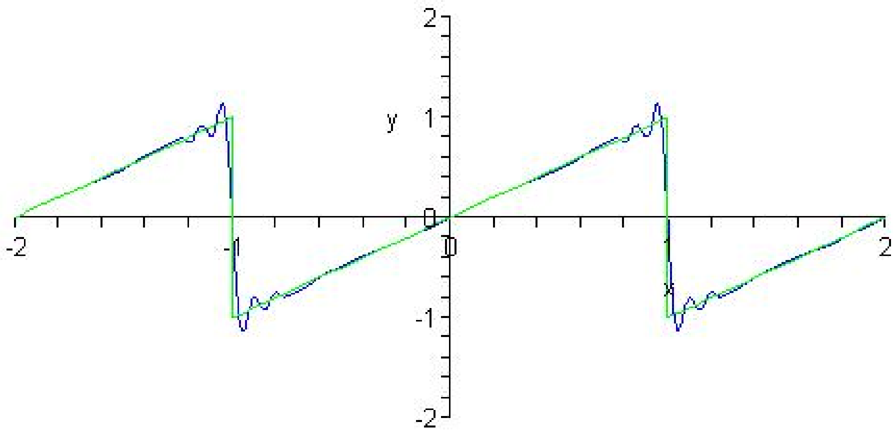

Figure 6: 20th order Fourier approximation for f(x)=x

3 FOURIER SERIES FOR DISCRETE DATA

In this section we derive the discrete Fourier transform and the inverse discrete Fourier transform. We first study concepts of Kernel and the Dirichlet Kernel in particular, to arrive at the required results.

3.1 The Kernel and Dirichlet Kernel

We first look at the concept of Kernel. Consider the integral,

$$\int_{-\pi}^{\pi} f(y)k(x-y)dy \tag{32}$$

For an integral of this form, we call $k(..)$ a kernel. Notice that the above integral defines a funtion on x. So any kernel actually creates a map from functions on $[-\pi, \pi]$ to itself.

We now look at a particular type of Kernel called the Dirichlet Kernel of order N. This is defined as,

$$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + \ldots + \cos Nx \tag{33}$$

Since we are dealing with the cosine function we know that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x)dx = 1$$

Using Euler's formula for complex numbers we can show that,

$$D_n(x) = \begin{cases} N + \frac{1}{2}, & x = 2\pi m, m \in \mathbb{Z}, \\ \frac{\sin(x(N+\frac{1}{2}))}{2\sin(\frac{x}{2})}, & \text{otherwise.} \end{cases} \tag{34}$$

We now see how the Dirichlet Kernel applies to the Fourier Series defined earlier,

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(u)D_N(x-u)du \\
&= \int_{-\pi}^{\pi} f(u)\left\{\frac{1}{2} + \int_{n=1}^{N} \cos n(x-u)\right\} du \\
&= \frac{1}{2}\int_{-\pi}^{\pi} f(u)du + \sum_{n=1}^{N}\int_{-\pi}^{\pi} f(x)[\cos nx \cos nu + \sin nx \sin nu]du \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx
\end{aligned} \tag{35}$$

which is the Fourier series formula where a_n and b_n are Fourier coefficients. We now look at a funtion $R_N(x)$ which will help measure error in the Fourier series approximation of a funtion $f(x)$.

$$\begin{aligned}
R_N(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) - f(x-t)D_N(t)dt \\
&= \int_{-\pi}^{\pi} f(x)D_N(t)dt - \int_{-\pi}^{\pi} f(x-t)D_N(t)dt \\
&= f(x).(1) + \int_{\pi+x}^{\pi-x} f(u)D_N(x-u)du \\
&= f(x) - \int_{-\pi}^{\pi} f(u)D_N(x-u)du \\
&= f(x) - S_N(x)
\end{aligned} \tag{36}$$

Where $S_N(x)$ is the Fourier approximation as in equation 35.

3.2 Discrete Fourier Transform

Earlier we defined the Fourier series in terms of trignometric functions in equation 18. Euler's formula expresses the exponential funtion in terms of the trignometric functions and complex

i,

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{37}$$

Using this identity, we can express the Fourier series formula is terms of the complex exponential function and coefficients c_k as follows,

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=0}^{\infty}(a_n\cos(nx) + b_n\sin(nx)) \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}} c_k e^{-ikx} \end{aligned} \tag{38}$$

where,

$$ck = \begin{cases} a_k + ib_k, & k > 0, \\ a_k - ib_k, & k < 0, \\ a_0, & k = 0 \end{cases}$$

The function f is defined on the Torus as described earlier. The coefficients c_n however, are defined on the intgers. We can obtain the entire real number sequence by integer shifts of 2π on the torus. We obtain the following relations,

$$\bigcup_{k\in\mathbb{Z}}(\pi + 2\pi k) = \mathbb{R} \tag{39}$$

$$\bigcup_{k\in\mathbb{T}}(x + 2\pi\mathbb{Z}) = \mathbb{R} \tag{40}$$

We use the complex number expression of the Fourier series in order to define the discrete Fourier transform. The set $\{a_k\}_{k\in\mathbb{Z}}$ in a set of complex numbers where each$\{a_k\}$ is a complex number. The discrete Fourier transform of $\{a_k\}_k$ is defined as,

$$\hat{a}(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi} \tag{41}$$

For the coefficients a_k we will assume that conditions for ℓ^2 are satisfied that is the following sum is convergent,

$$\sum_{k\in\mathbb{Z}} |a_k|^2 < \infty \tag{42}$$

Now that the discrete Fourier transform is defined, we will try to extract the coefficients a_k from the transform $\hat{a}(\xi)$. Consider coefficients c_k such that,

$$
\begin{aligned}
c_k &= \frac{1}{2\pi}\int_{\pi} \hat{a}(\xi)e^{ik\xi}d\xi \\
&= \frac{1}{2\pi}\int_{\pi} (\sum_{k\in\mathbb{Z}} a_k e^{-ik\xi})e^{ik\xi}d\xi \\
&= a_k
\end{aligned}
\tag{43}
$$

3.3 Dicrete Finite Fourier Transform

Consider the space $\mathbb{R}^N$ as depicted below.

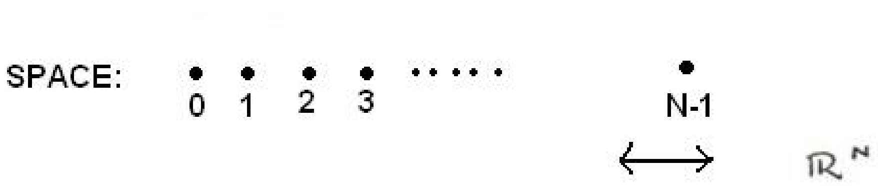

Figure 7: Discrete space $\mathbb{R}^N$

Values m,n are part of this space. We will now try to understand the concept of 'mod' for these two values. If $m = 3$, $n = 2$ and $N = 4$ we can clearly see that,

$$
\begin{aligned}
N\mathbb{Z} + (m+n) &= 4\mathbb{Z} + 5 \\
&= 4\mathbb{Z} + 1
\end{aligned}
$$

Thus for this given data, the mod is defined as $3+2 = 1(\text{mod } 4)$. Thus for $(m+n)(\text{mod } N)$ we just subtract N a fixed number of times from m+n if we go out of the space. Similarly we define $\mathbb{Z}$ mod $N\mathbb{Z}$ and use the notation $\mathbb{Z}/N\mathbb{Z}$ for it. We notice that,

$$
\begin{aligned}
\bigcup_{j=0}^{N-1}(N\mathbb{Z}+j) &= \mathbb{Z} \\
\bigcup_{j\in\mathbb{Z}/N\mathbb{Z}}(N\mathbb{Z}+j) &= \mathbb{Z}
\end{aligned}
\tag{44}
$$

We now define a value ω_N as the Nth order root of unity, which means, $\omega^N = 1$. Consider,

$$
\omega = e^{\frac{2\pi i}{N}} = \cos\frac{2\pi}{N} + i\sin\frac{2\pi}{N} \tag{45}
$$

Observe that,

$$z^N - 1 = (z-1)(z^{N-1} + z^{N-2} + \ldots + z^2 + 1) \tag{46}$$

$$1 + \omega_N + \ldots + \omega_N^{N-1} = 0 \text{ (By substitution } z = \omega_N) \tag{47}$$

Thus, as an example, if N=3,

$$1 + e^{i\frac{2\pi}{3}} + e^{i\frac{4\pi}{3}} = 0 \tag{48}$$

or when N=4 we have,

$$i + i^2 + i^3 + 1 = 0 \tag{49}$$

We notice that the a function defined on $\mathbb{Z}/N\mathbb{Z}$ can be thought of as an element of $\mathbb{R}^N$. We now define the Discrete Finite Fourier Transform for a function f defined on $\mathbb{Z}/N\mathbb{Z}$ where $f \in \mathbb{R}^N$. $f(m)$ is the value of the function f at m where, $m \in \mathbb{Z}/N\mathbb{Z}$. The discrete finite Fourier transform $\hat{f}(k)$ is,

$$\hat{f}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) e^{\frac{-2\pi i}{N}kn} \tag{50}$$

The inverse Discrete Finite Fourier Transform is defined as,

$$\check{f}(n) = \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} f(k) e^{\frac{2\pi i}{N}kn} \tag{51}$$

In order to see that the finite Fourier transform and its inverse as described above makes sense, we apply the inverse transform to a transform on $f(m)$ and check to see if we can get back the same function,

$$\begin{aligned}
\check{\hat{f}}(m) &= \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \hat{f}(k) e^{\frac{2\pi i}{N}km} \\
&= \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} (f(n) e^{\frac{-2\pi i}{N}kn}) e^{\frac{2\pi i k m}{N}} \\
&= \sum_{n \in \mathbb{Z}/N\mathbb{Z}} f(n) \frac{1}{N} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} e^{\frac{2\pi i k(m-n)}{N}} \\
&= \begin{cases} f(m), n = m, \\ 0, n \neq m. \end{cases}
\end{aligned} \tag{52}$$

Thus, $\check{\hat{f}}(m) = f(m)$.

3.4 APPLICATION

We will now apply the finite Fourier transform earlier obtained to image processing and compression. An image can be understood as a two dimensional set of discrete data. An image comprises of pixels and in the case of a black and white image, each pixel has a specific intensiy value. These values represent the discrete data that we will process.

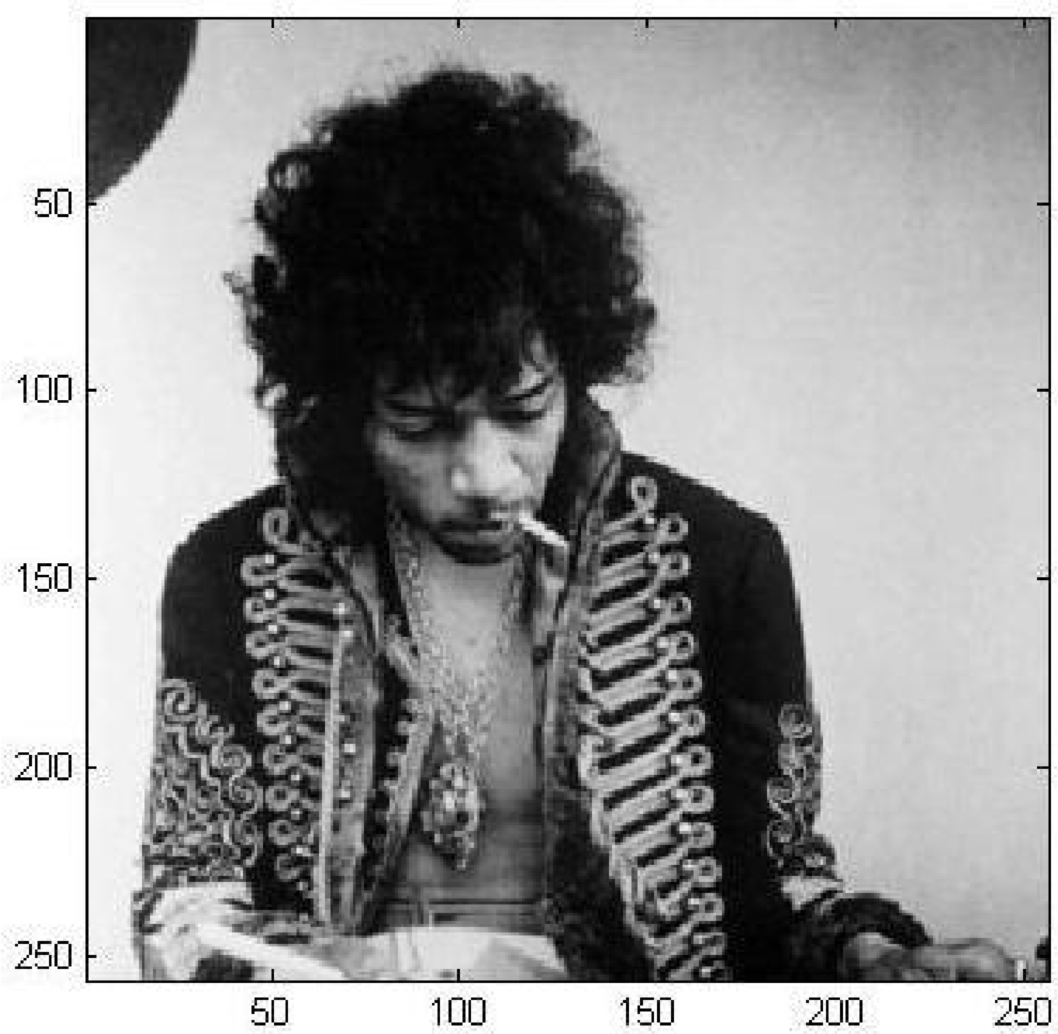

Figure 8: Original Image

Since the image comprises a 2 dimensional set of data we will use the 2-D Discrete finite Fourier transform and its inverse, formulae for which are,

$$\hat{x}(l,m) = \frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{N} x_{j,k} e^{\frac{-2\pi ijl}{N}} e^{\frac{-2\pi ikm}{N}} \tag{53}$$

$$\check{x}(j,k) = \frac{1}{N}\sum_{l=1}^{N}\sum_{m=1}^{N} x_{l,m} e^{\frac{2\pi ijl}{N}} e^{\frac{2\pi ikm}{N}} \tag{54}$$

In these formulae, the co-ordinates (l, m) and (j, k) denote specific pixels on the picture and N is the number of pixels on the edge of a square picture. Matlab code is used in order to apply the discrete finite Fourier transform to an image. After this, the transform is manipulated in order to retain a certain fraction of transform elements and the rest are set to 0. Following this the modified transform is converted back to the image using the inverse discrete finite Fourier transform. The image we will use here is of size 256x256 meaning that it is a square image with 256 pixels to an edge. The original image with no transformation applied is shown in Figure 8.

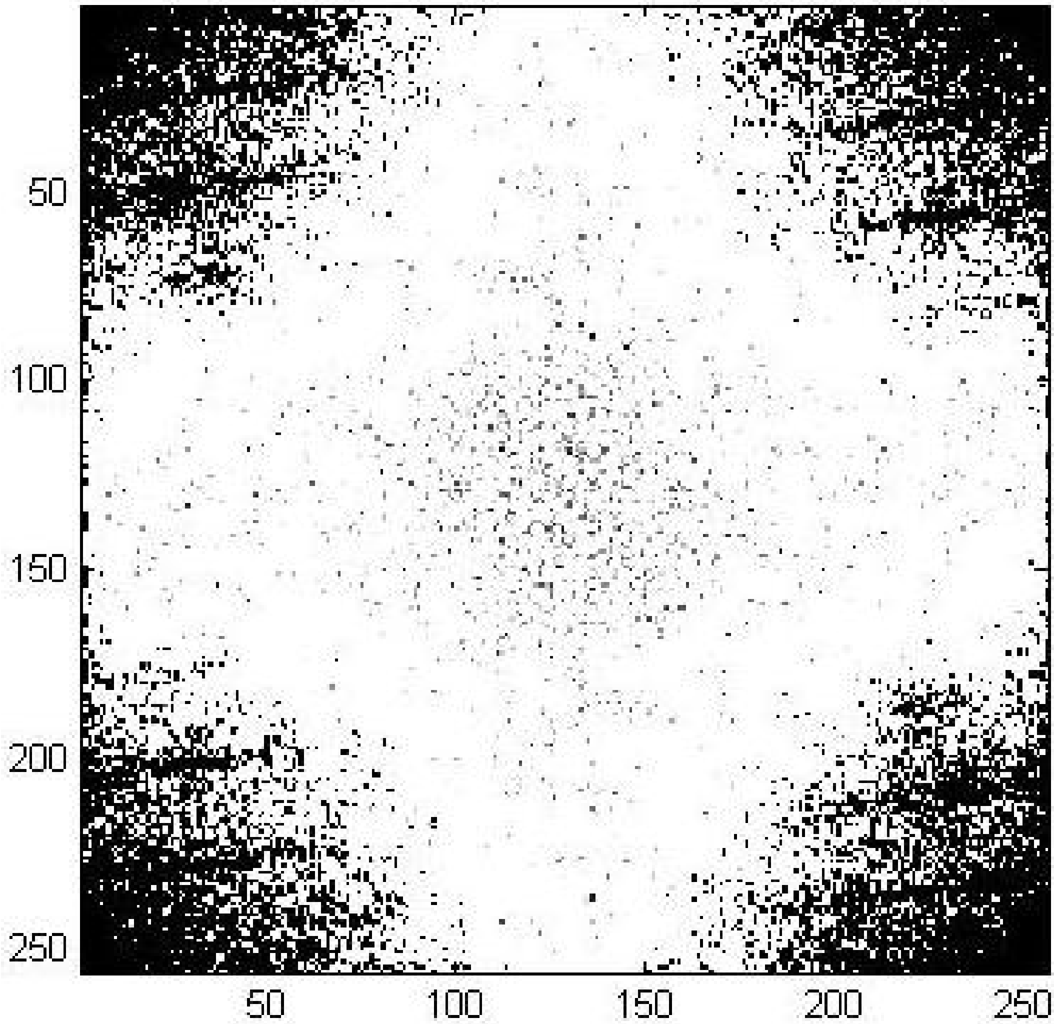

Figure 9: Image of the transform

Image 9 is the output image after the Discrete finite Fourier transform is applied to the original image. Notice that the black area of the image represents the Pixels that were set to 0 during the compression. We will now apply the inverse discrete finite Fourier transform to image 9 in order to obtain our compressed image.

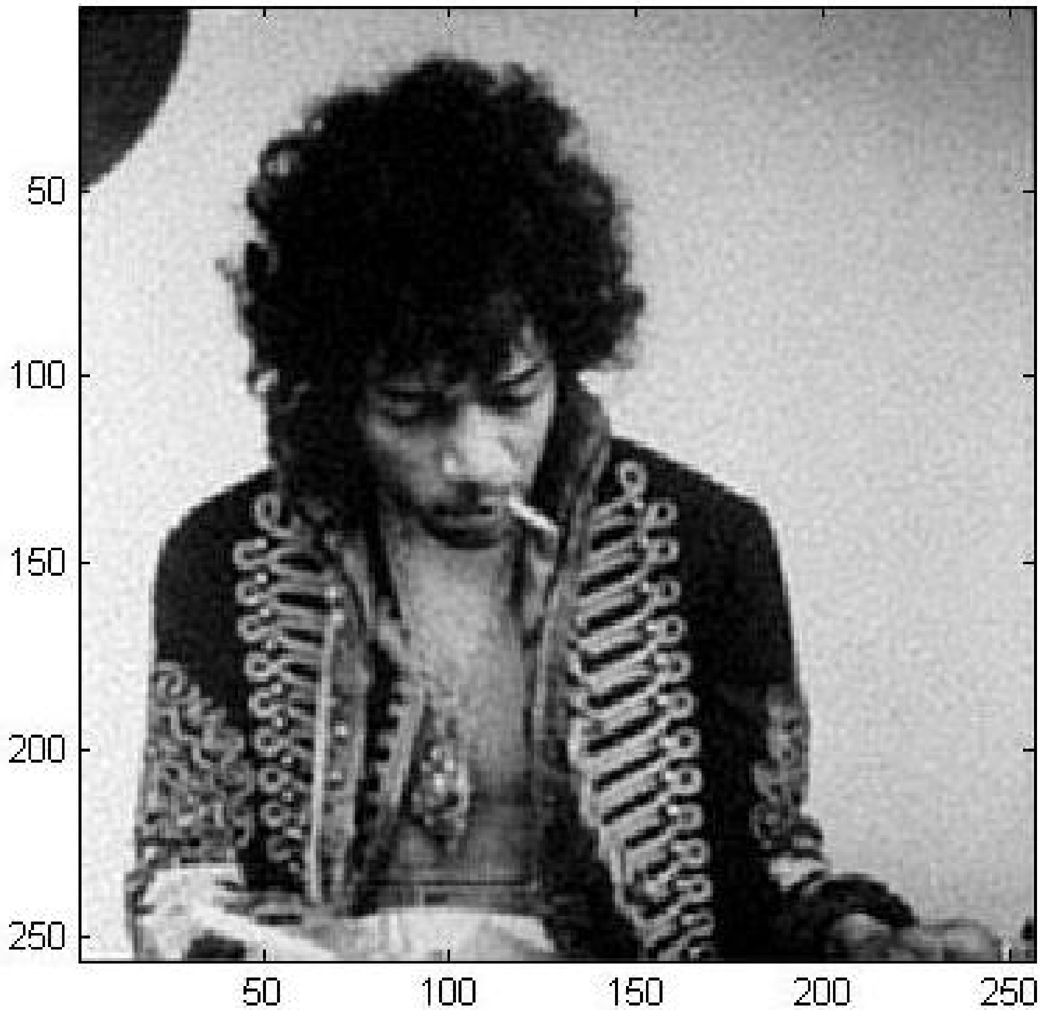

Figure 10: Image of the inverse transform - compressed

Notice that in the images, the transformed image achieves a significant compression of about 5.12 : 1 without significant loss in image quality. It is thus possible to achieve smaller compressions using the discrete finite Fourier transform (such as 2:1) with almost no loss in image quality[3].

[3]Imaging done using The MathWorks Matlab 7.

References

[1] Kirkwood, James R., *An Introduction to Analysis*, PWS-KENT, (1989).

[2] Mallat, Stephane, *A Wavelet Tour of Signal Processing*, Academic Press, (1999).

[3] Penny, Richard C., *Linear Algebra- Ideas and Applications*, Wiley, (2004).