PROJECTILES APPROACHING OR RECEDING - A UNIQUE PARADOX

1 INTRODUCTION

This project involves the use of basic equations of motion for a projectile assuming constant vertical acceleration. A unique feature of this projectile motion is studied here, which is the distance between the point of projection the projectile and the position of the projectile at various times during its flight. The variation of this distance is studied using the equations of motion and some basic calculus and very interesting and unexpected results are obtained. The various results and equations are graphed in order to obtain a better visual understanding of what is actually going on. The question that we will pose now will be answered in this paper - 'Is a traveling projectile projected at some known angle of projection and with a given velocity approaching us or receding away from us?' Section 5 of this project looks at the important properties of the 45° angle of projection and its relation to other project results.

2 BACKGROUND

In this paper, we will consider basic projectile motion, with a projectile launched from the surface at a known angle, i.e. the angle between the initial velocity vector of the projectile and the horizontal is known. The x direction is defined to be the horizontal direction and the y direction is defined to be the vertical direction. All velocities are broken down into x and y components and motion in these directions is independently analyzed in order to obtain the parametric equations of motion[1] for the parabolic path.

Consider a projectile launched at an angle of θ with the x direction with an initial velocity of projection of magnitude v. Breaking this velocity into rectangular components we have initial velocities of $v\cos\theta$ in the x direction and $v\sin\theta$ in the y direction.

[1] Hibbeler Dynamics, 10^{th} edition pg. 38

Analyzing motion in the x direction, we see that we have a constant velocity in this direction with no acceleration. Thus, distance traveled in the x direction, X, is given by,

$$X = (v \cos\theta)t \tag{1}$$

where the parameter t in this equation represents the time elapsed since the time of projection ($t = 0$) at the instant of observation.

Analyzing motion in the y direction, we see that a constant acceleration exists in this direction, which is the acceleration due to gravity. From the equations of motion, the distance traveled in this direction, Y, is given by,

$$Y = (v \sin\theta)t - \frac{1}{2}gt^2 \tag{2}$$

Notice that both these equations are defined using the time parameter t. We now define a new variable t_f which represents the time of flight for the projectile. In order to obtain this we observe that $t = t_f$ when $Y = 0$. Hence from equation 2 we have,

$$t_f = \frac{2v \sin\theta}{g} \tag{3}$$

From equations 1 and 2 we can eliminate parameter t and obtain an equation for the path of the projectile in variables x and y,

$$y = x \tan\theta - \frac{1}{2}\frac{g}{v^2 \cos^2\theta}x^2 \tag{4}$$

We can graph this equation on x and y axes and obtain the parabolic path of the projectile as shown in Figure 1(page 3). Notice also that we intend to study the variation of distances r_1, r_2, $r_3 \ldots$, as drawn on the graph, as the core aim of the project.

2.1 Restrictions

As one will notice, we have placed certain restrictions on the projectile motion we look at for purposes of clarity and simplicity. Firstly, the initial and final points of the projectile motion have the same y coordinate, that is $y = 0$. Also we restrict ourselves to constant vertical acceleration and drag free motion. In addition, we require that the velocity vector of projection is known, that is, we know the angle of projection and the magnitude of velocity.

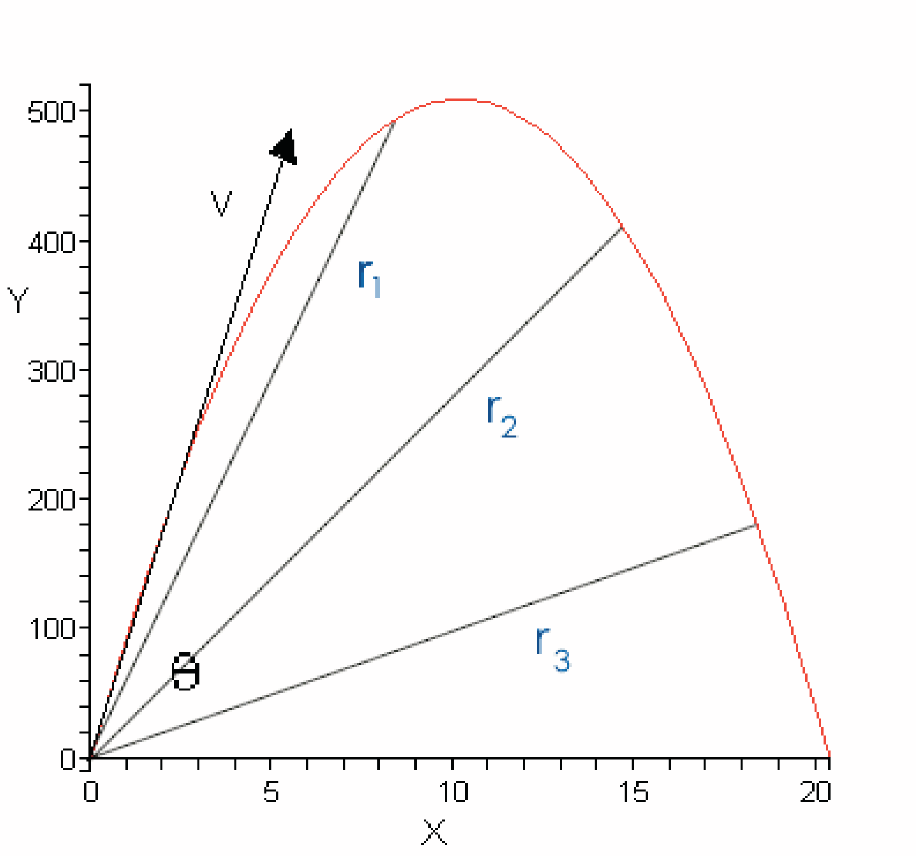

Figure 1: Parabolic projectile path

3 PROBLEM SOLUTION

In this section we do the mathematical work necessary to answer the question set out in the introduction, i.e., 'Is a traveling projectile approaching us or receding away from us?' In order to do this we must first define a quantity 'D' which is the point-to-point distance between the point of projection of the projectile at $t = 0$ and the point of observation at some time t. From the x and y distances in equations 1 and 2, using Pythagoras theorem we have,

$$\begin{aligned} D &= \sqrt{x^2 + y^2} \\ &= \sqrt{v^2t^2 \cos^2\theta + v^2t^2 \sin^2\theta + \frac{1}{4}g^2t^4 - (v\sin\theta)gt^3} \\ &= \sqrt{v^2t^2 + \frac{g^2t^4}{4} - vgt^3 \sin\theta}. \end{aligned} \tag{5}$$

In formula 5 above, $(v = constant) \geq 0$ and $\theta \in [0, \frac{\pi}{2}]$. Now that we have a formula that provides us with the distance of the projectile from the point of projection, we are interested in studying how this distance varies, particularly by finding points of Maxima or Minima for this distance equation. Since maximizing the function D is similar to maximizing the function D^2, we define an $F = D^2$. We look at the case where we have a known constant angle of projection θ and a known initial velocity magnitude v, and find critical points for F by setting $\frac{dF}{dt} = 0$. This leads to,

$$t[g^2t^2 - (3vg\sin\theta)t^2 + 2v^2] = 0 \tag{6}$$

Thus, for this equality to hold true, either $t = 0$, in which case no projection takes place, or,

$$\begin{aligned} g^2t^2 - (3vg\sin\theta)t^2 + 2v^2 &= 0 \\ t^2 - (3\frac{v}{g}\sin\theta)t + 2\frac{v^2}{g^2} &= 0 \end{aligned} \tag{7}$$

Solving the above quadratic equation for $t_{max/min}$,

$$t_{max/min} = \frac{3v}{2g}\left\{\sin\theta \pm \sqrt{\sin^2\theta - \frac{8}{9}}\right\} \tag{8}$$

From the equation above, it is clear that for real values of $t_{max/min}$ to exist,

$$\begin{aligned} \sin^2\theta - \tfrac{8}{9} &\geq 0 \\ \sin^2\theta &\geq \tfrac{8}{9} \\ \sin\theta &\geq \pm\tfrac{\sqrt{8}}{3}, \text{ but since } \theta \in [0, \tfrac{\pi}{2}], \\ \sin\theta &\geq \tfrac{\sqrt{8}}{3} \text{ OR } \boldsymbol{\theta \geq \sin^{-1}\tfrac{\sqrt{8}}{3} \approx 70.5287\,^\circ}. \end{aligned} \tag{9}$$

Therefore we notice that for critical points to exist on a distance from projection vs. time of flight graph, the angle of projection must be above this particular angle of $70.5287\,^\circ$ obtained in equation 9. We will call this angle the critical angle. It is also important to notice that the critical angle obtained is independent of projection velocity or vertical acceleration magnitudes.

We have equation 3 for the total time of flight for the projectile t_f and equation 9 for time of critical point $t_{max/min}$. The ratio $\frac{t_{max/min}}{t_f}$ is a fraction expressing the x position of maximum/minimum distance point w.r.t. total horizontal (x direction) flight distance or range[2].

$$\frac{t_{max/min}}{t_f} = \frac{3}{4}\left\{1 \pm \frac{\sqrt{\sin^2\theta - \frac{8}{9}}}{\sin\theta}\right\} \tag{10}$$

4 GRAPHICAL RESULTS

It is important to graphically realize some of the results obtained in the previous section. The first graph[3] we look at comes directly from equation 10. We graph the positive version of the $\pm$ equation.

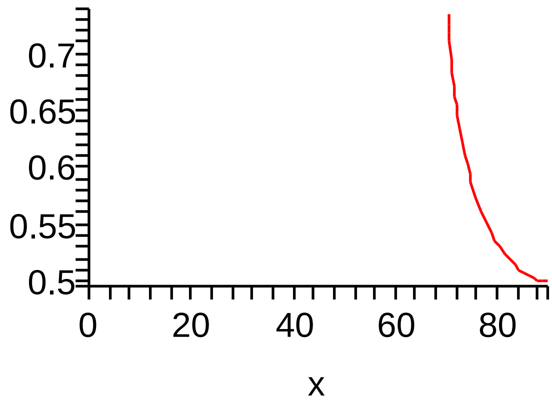

Figure 2: Positive position ratio vs. angle

From the graph we can clearly see that a position for the max/min distance exists only for an angle above the angle derived in equation 9. To interpret this graph, for an angle of projection of 80°, the max/min point exists when the projectile has traveled 53% of its horizontal distance, i.e. the point (80,0.53) on the graph.

We now graph the complete equation 9 with the time ratio on the y axis and projection angle on the x axis.

[2] This is because horizontal range is directly proportional to t_f in equation 1

[3] All graphs constructed using Maplesoft MAPLE 9.5

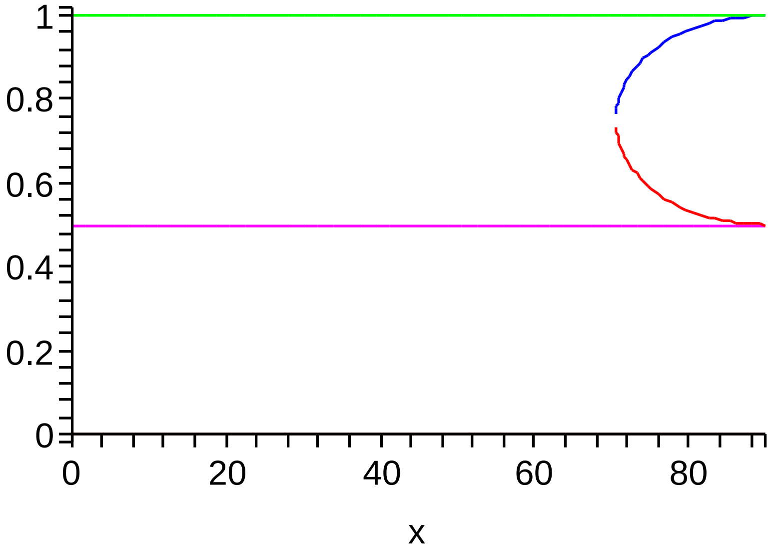

Figure 3: Positive position ratio vs. angle

The horizontal lines on this graph are asymptotes representing $y = .5$ and $y = 1$ respectively. It is quite interesting to note that at angles above the critical angle, two critical points exist for the distance formula. We can interpret this as two locations where distance of the projectile maximize and minimize along the path of the projectile which is quite an interesting result. A better understanding of this result can be seen in the graphs to follow. We also see that at a projection of 90° the point at which we achieve maximum distance occurs at 50% of our flight time which conforms to our intuition of vertical projection, where maximum distance exists at zero velocity and half of total flight distance and time.

To better understand this graph we take the following example. If we observe the $x = 75°$ vertical line, we will see that it will intersect the curve at two points, i.e., $y = 0.58$ and $y = 0.91$. This can be interpreted as, with an angle of projection of 75° the distance from the point of projection to the projectile maximizes or minimizes at 58% and 91% of total flight time.

4.1 Particular Cases

We will now observe distance from origin vs. time of flight graphs for projectiles projected at 3 particular angles. The first angle we observe is projection below critical angle. Let us consider $\theta = 60°$ for example.

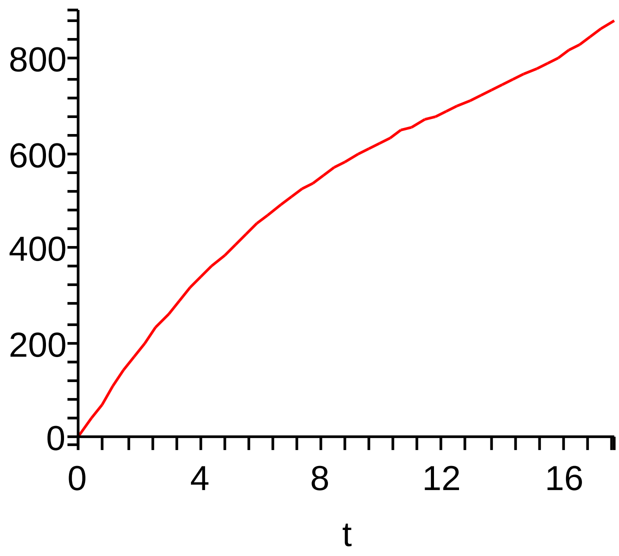

Figure 4: Distance vs. time for $\theta = 60°$

As we can see, no critical points exist on this graph and the distance from projection keeps increasing with time. A similar graph will be observed for all $\theta < 70.5287°$. In figure 5 (page 8) we observe what happens when we chose projection angle to be equal to the critical angle, i.e., $\theta = 70.5287°$.

As clear from this graph, a single critical point is observed indicating at a projection equal to the critical angle. Inflection takes place on the graph at a critical point. Lastly, in figure 6 (page 8) we will observe the case where the angle of projection is greater than the critical angle, say, $\theta = 78°$.

Figure 6 is an interesting graph to observe because as expected we see both a maximum and a minimum point on this graph. Hence for projections greater than the critical angle, the projectile retreats from the point of projections, approaches it, and retreats again.

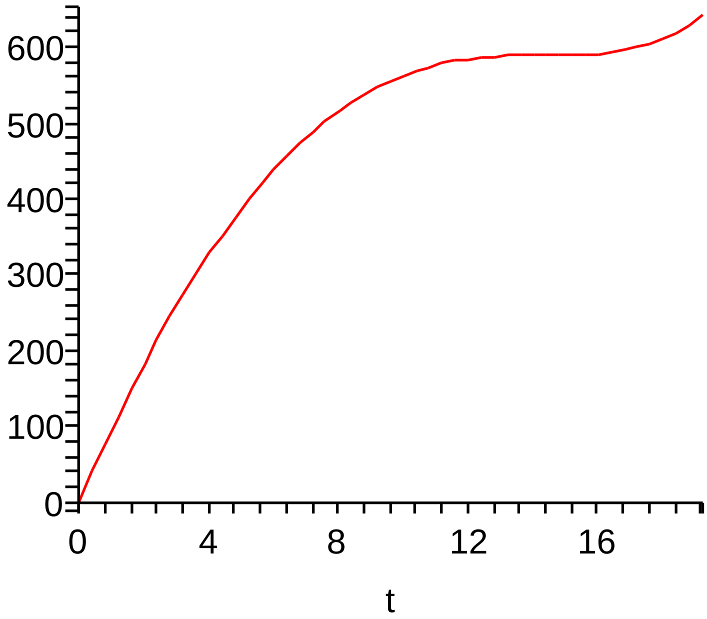

Figure 5: Distance vs. time for $\theta = 70.5287°$

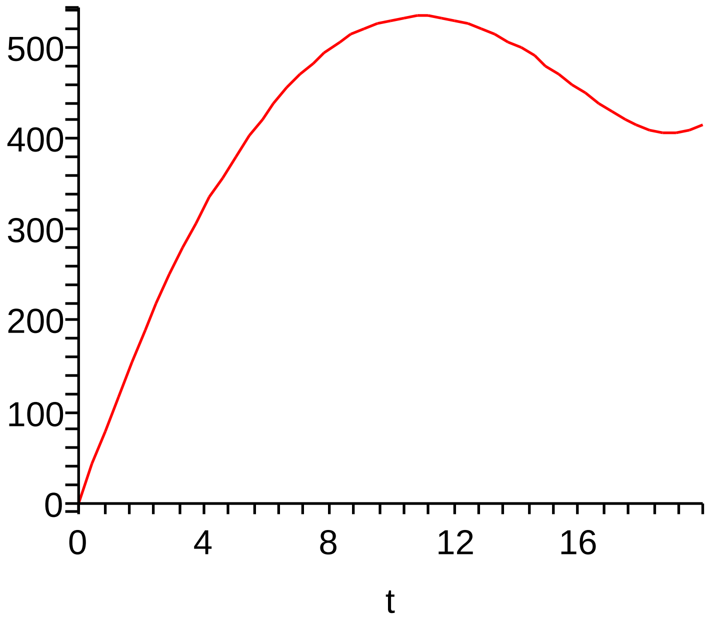

Figure 6: Distance vs. time for $\theta = 78°$

It is also important to note that is each of the graphs above, the velocity and acceleration act only as a scaling factors for the distance on the y axis and have no direct effect on the properties of the graph itself.

5 45° PARADOX

It is especially important to consider the 45° angle of projection case. It is known that for a projectile to have a maximum horizontal range on projection from the point of projection, the angle of projection must be 45°. This in turn is the expected critical angle for any variation in behavior, like the distance variation observed in this paper, because it conforms to intuition since it is halfway between 0° and 90°. We will now compare the variation of the distance from the point of projection with time for various angles.

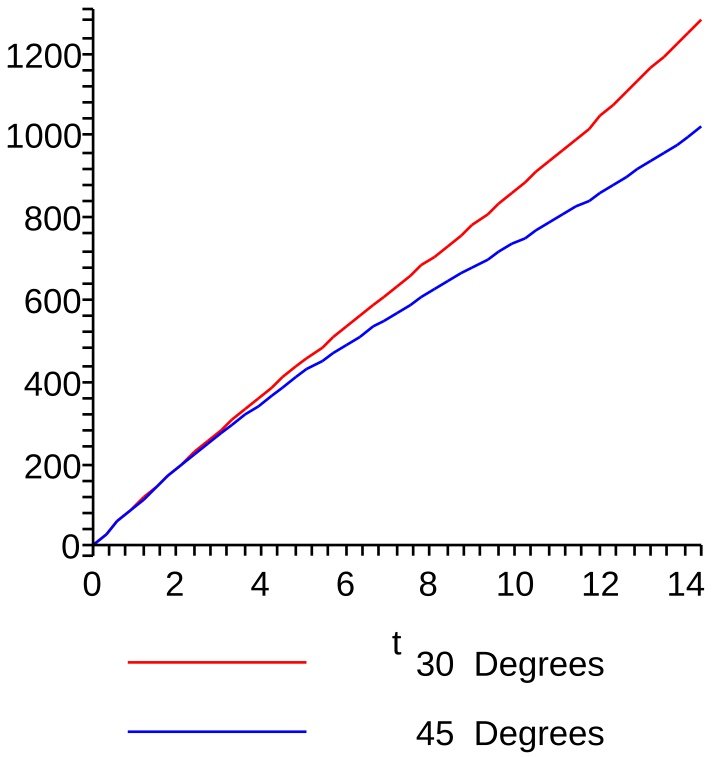

Figure 7: Distance vs. time for $\theta = 45°$ and $\theta = 30°$

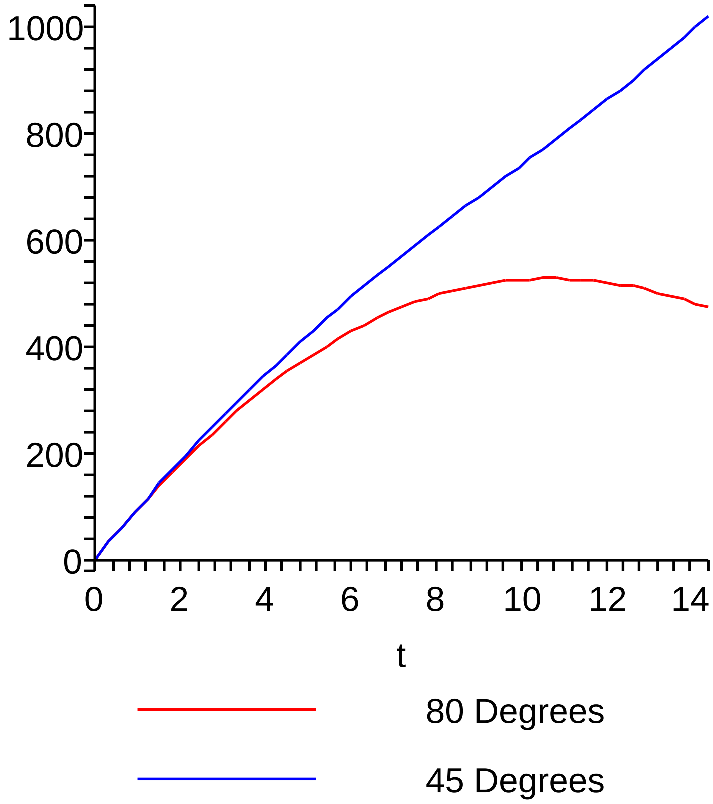

Figure 8: Distance vs. time for $\theta = 45^\circ$ and $\theta = 80^\circ$

As we see in these two figures, the 45° angle of projection represents no critical change in graph behavior, but the 80° ($> 70.5287^\circ$) angle shows significant deviation in behavior. Thus, paradoxically, the critical angle derived is a special case angle for projectile motion that is seemingly random and very much different from the expected 45° whole number angle.

5.1 90° angle of projection

The 90° angle of projection is another important special case to look at. We expect that if we construct a graph between distance from the point of projection and time of flight for a projection angle of 90°, the maximum distance should be observed at $\frac{1}{2}$ flight time, and the final distance must be 0 as the projectile returns back to point of launch. We observe this in figure 9,

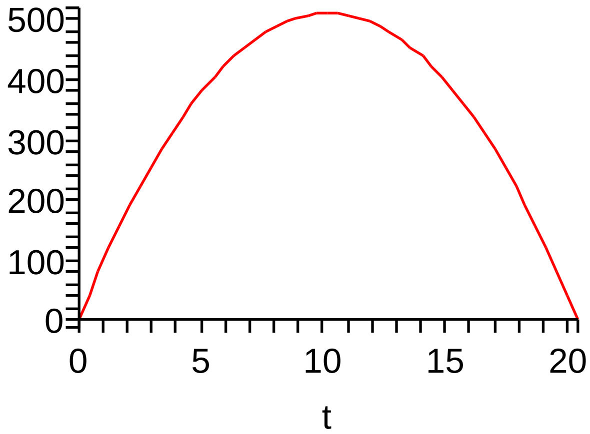

Figure 9: Distance vs. time for $\theta = 90°$

6 CONCLUSIONS

In this experiment we observed the properties of the distance between the point of projection for a projectile and the projectile itself, for the duration of its flight. The following were the key results,

- A critical angle of 70.5287° was derived for projectile motion above which the projectile reaches a maximum/minimum distance from the point of projection. This angle is independent of projection velocity and acceleration due to gravity.
- For projections above the critical angle, two critical points are observed. Therefore, the projectile retreats away from us, approaches us for a period of time, and then retreats again. This phenomenon is quite different from what we would intuitively expect.
- The critical angle derived is an example of an interesting paradox when compared to a 45° angle of projection which is expected to be an angle at which critical changes happen during projectile motion.

Darren Pais, Engineering Mechanics - Dynamics (ES-P-310) Final Project

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